Approximation Algorithms for Planar Matching*

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1. Introduction

Approximation algorithms for NP-Complete optimization problems have been widely studied, but rarely for problems in P. Given an optimization problem in P, a question one asks is whether known approximations for NP-Complete problems can be used as subroutines in approximately solving this problem. In any attempt at approximating problems in P, we can require (1) that the algorithm be almost linear or run in $O(n (\log n)^k)$ time, and (11) that the relative error of the approximation be vanishingly small. Condition (1) facilitates approximate solution of much larger instances using currently available computers, as opposed to, say, a $\Theta(n^2)$ exact algorithm. Condition (11) helps specify arbitrarily small relative errors, say, 1%.

Here we exhibit two approximation algorithms for the problem of finding a maximum weighted matching in a planar graph, for which the best exact algorithm runs in $O(n^{3/2} \log n)$ time [3].

One runs in $O(n f(n) \log n^2)$ time, and uses divide-and-conquer based on the separation of $f(n)$-outerplanar graphs as defined in [1]. Here $1/f(n)$ is the relative error demanded of the approximation, and hence this is a fully polynomial time approximation. For instance, setting $f(n) = \log n$, we get relative error asymptotically tending to zero.

Another approximation algorithm runs in $O(nc f(n))$ time. Here $c$ is a constant [1]. For instance, setting $f(n)$ equal to $\log^* n$, we get an almost-linear-time approximation with vanishing relative error.

*Research supported by NSF grant DCR-8402045
2. Weighted Matching is As Easy As Bounded-degree Weighted Matching

First, we show that the problem of finding a maximum weighted matching (hereafter called MWM's) in graphs of bounded degree is as hard as finding MWM's in general graphs; that is, we show how, given an instance $G$ of the MWM problem, how to construct a graph $G'$ of degree $\leq 3$, such that a MWM in $G'$ can be transformed in linear (in size of $G$ and $G'$) time to a MWM in $G$. Thus bounded degree does not help. For a similar reduction for the vertex cover problem, see [2].

The reduction takes each vertex $v$ of degree $d_v \geq 2$ in $G$ and replaces it by a cycle $C_v$ of length $2d_v + 1$, where the edges on the cycle are given a weight $M$ larger than the maximum edge in $G$, and where the original $d_v$ edges connected to $v$ in $G$ are now connected to $d_v$ mutually non-adjacent vertices in the cycle (If the graph is planar, we will later insist that the cyclic order of the edges be the same when reconnection of edges to the cycle occurs). See figure 1.

Lemma 1: Let $X'$ be a MWM in $G'$. Let $X'_v$ be the edges of $X'$ incident on vertices of $C_v$. Then $|X'_v \setminus E(G)| = 0$ or 1.

Proof: Let $m_v = |X'_v \setminus E(G)|$. If $m_v = 0$, then clearly $|X'_v| = d_v$. If $m_v = 1$, then clearly $|X'_v| = d_v + 1$. Suppose it were true that $m_v \geq 2$; clearly, $|X'_v| \leq d_v + 1$; however, we can get a better matching by simply omitting all but the maximum edge from $X'_v \cap E(G)$ and augmenting the same number of edges (each of larger weight) from the cycle $C_v$ itself, contradicting the fact that $X'$ is maximum; Therefore, $m_v$ cannot exceed 1. \(\square\)

Lemma 2: A MWM $X$ in $G$ can be directly obtained from a MWM $X'$ in $G'$ by contracting the cycles introduced in obtaining $G'$ from $G$. That is, $X' \cap E(G)$ is a MWM in $G$.

Proof: (Here we will speak of the edges of the original graph as the preimage of the