Chapter 4

Foundations of a Lie algebraic theory of geometrical optics

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ABSTRACT: We present the foundations of a new Lie algebraic method of characterizing optical systems and computing their aberrations. This method represents the action of each separate element of a compound optical system—including all departures from paraxial optics—by a certain operator. The operators can then be concatenated in the same order as the optical elements and, following well-defined rules, we obtain a resultant operator that characterizes the entire system. These include standard aligned optical systems with spherical or aspherical lenses, models of fibers with polynomial z-dependent index profile, and also sharp interfaces between such elements. They are given explicitly to third aberration order.

We generalize a previous result on the factorization of the optical phase-space transformation due to a refraction interface. We also present a group-theoretical classification for aberrations of any order of systems with axial symmetry, applying it to the problem of combining aberrations; new insights are thus provided on the origin and possible correction of these aberrations. We give a fairly complete catalog of the Lie operators corresponding to various simple optical systems. Finally, there is a brief discussion of the possible merits of constructing a computer code, RAYLIE, for the Lie algebraic treatment of geometric ray optics.

4.1 Introduction

Let us consider the optical system illustrated schematically in Figure 1: a ray originates at the general initial point $P^i$ with spatial coordinates $r^i$ and moves in an initial direction specified by the unit vector $\hat{s}^i$. After passing through an optical device, the ray arrives at a final point $P^f$ with coordinates $r^f$ and in a direction specified by the unit vector $\hat{s}^f$. The fundamental problem of geometrical optics is: given the initial quantities $(r^i, \hat{s}^i)$ and a specification of the optical device, to determine the final quantities $(r^f, \hat{s}^f)$. We may search for the design of an optical device such that the relation between families of initial and final rays have various desired properties, such a focusing, Fourier transformation, or other operations.
The purpose of this chapter is to provide the foundations for a Lie algebraic approach to the problem of characterising optical elements and optical systems \[1\]. In this approach, we represent the action of each separate element of a compound optical system, including all departures from paraxial gaussian optics, by a corresponding operator. These operators can be concatenated to obtain the resultant operator that characterizes the entire system. Lie algebraic methods provide an operator extension of the matrix methods of gaussian (paraxial) optics to the general case \[2\].

We believe that the Lie algebraic approach simplifies the calculation of aberrations, and may facilitate their correction. Moreover, this approach is ideally suited to machine computation. Finally, the operator methods developed for geometrical optics may also have extensions to wave optics. This is the subject of ongoing research.

The organization of this chapter is as follows. In Section 2 we show that every optical system gives rise to —and is characterized by— a symplectic map. In Section 3 we present the necessary Lie algebraic tools, and show the way in which symplectic maps can be written as products of Lie transformations. Section 4 describes the computation of Lie transformations for continuous systems, and Section 5 describes the treatment of discontinuous interfaces between two such media. In Section 6 we relate these Lie transformations to paraxial optics and describe aberrations as classified into symplectic group multiplets. We apply this classification in Section 7 to the problem of combining aberrations. In a final section we describe briefly the possible merits of constructing a computer code, RAYLIE, for the Lie algebraic treatment of geometric ray optics.

4.2 Optical symplectic maps

In this section we summarize the Fermat–Hamilton formulation of geometrical optics. This will provide the basis for other chapters in the present volume, and is particularly well suited for the introduction of the Lie algebraic methods in the next section. The “black-box” optical device of Figure 1 will be now specified as a three-dimensional medium where an index of refraction function \(n(r) = n(x, y, z)\) is defined. A homogeneous medium corresponds to \(n(r) = \text{constant}\). In vacuum \(n = 1\), while for any other medium, \(n > 1\).

\[1\] This article gives a preliminary description of the use of Lie algebraic methods in geometrical optics.