The Boolean Hierarchy: Hardware over NP

Jin-yi Cai*
Lane Hemachandra #

Department of Computer Science
Cornell University
Ithaca, NY 14853

ABSTRACT

In this paper, we study the complexity of sets formed by boolean operations (∪, ∩, and complementation) on NP sets. These are the sets accepted by trees of hardware with NP predicates as leaves, and together form the boolean hierarchy.

We present many results about the boolean hierarchy: separation and immunity results, complete languages, upward separations, connections to sparse oracles for NP, and structural asymmetries between complementary classes. Some results present new ideas and techniques. Others put previous results about NP and D^P in a richer perspective. Throughout, we emphasize the structure of the boolean hierarchy and its relations with more common classes.

1. Introduction

NP has long been the collective obsession of computer scientists. Yet some have turned from her siren song to study less alluring classes. D^P, the closure of NP ∪ coNP under intersections, has recently become immensely fashionable [PY82][PW85][CM85].

In this paper, we study the natural completion of such structures—the closure of NP under boolean operations.

The boolean hierarchy has appeared in passing in many papers [PZ82][R85][W85].

Previou results note the location of the boolean hierarchy: BH ⊆ P^#P (in fact, a single

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call to a \#P oracle suffices) [PZ82] and, more recently, BH \subseteq PP [R85]. This paper broadly investigates the structure of the boolean hierarchy itself. A more detailed version of this paper, which includes full proofs, oracle constructions that we've omitted here, and some additional results, appears in [CH85].

This closure, the boolean hierarchy (BH), has a clear "physical" interpretation. The sets in the boolean hierarchy are exactly those representable by hardware over NP. Each boolean hierarchy language is accepted by a hardware tree connecting NP machines [Figure 1].

We present many results about the boolean hierarchy: separations and immunity results, complete languages, connections with upward separations and sparse oracles for NP, and structural asymmetries between complementary classes. Some results present new ideas and techniques. Others put previous results about NP and DP in a richer perspective. Throughout, we emphasize the structure of the boolean hierarchy and its relations with more common classes.

2. Definitions

Each level of the boolean hierarchy consists of sets represented by a certain fixed structure of boolean operators on NP sets. Figure 2 shows the form of the boolean hierarchy, as defined below.

\[ \text{NP}(0) = \text{P} \]
\[ \text{NP}(2i + 1) = \{ L_A \cup L_B \mid L_A \in \text{NP}(2i), L_B \in \text{NP} \} \]
\[ \text{NP}(2i + 2) = \{ L_A \cap L_B \mid L_A \in \text{NP}(2i + 1), L_B \in \text{NP} \} \]
\[ \text{BH} = \bigcup_{i > 0} \text{NP}(i) \]
\[ \text{coNP}(i) = \{ S \mid \bar{S} \in \text{NP}(i) \} \].

For example, \( \text{NP}(1) = \text{NP} \), \( \text{NP}(2) = \text{DP} = \{ L_1 \cap \overline{L_2} \mid L_1, L_2 \in \text{NP} \} \), \( \text{NP}(3) = \{ (L_1 \cap \overline{L_2}) \cup L_3 \mid \)