CONTAINMENT, SEPARATION, COMPLETE SETS, AND IMMUNITY OF COMPLEXITY CLASSES

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1. Introduction

Many of the central open questions of computational complexity have the form: given two complexity classes \( W_1, W_2 \), is \( W_1 \) contained in \( W_2 \) (containment) or of the form: given two complexity classes \( W_1, W_2 \) is it true that \( W_1 \neq W_2 \) (separation). For instance, the question whether \( \text{NPCP} \) and the question whether \( \text{NP} \) contains a deterministic superpolynomial time class are of the above form. Other important questions have to do with the notions of immune and simple languages. For example, can \( \text{NP-complete} \) languages be \( \text{NP-simple} \)?

In this paper we study relations between containment, separation, complete sets, immunity and simplicity of complexity classes. The purpose is to obtain a good understanding of the structure of complexity classes. In recursion theory, without the ten year development of a rich theory regarding the r.e. sets, it would be hard to imagine the invention of the Freidberg-Muchnik priority method which resolved the Post's problem. In complexity theory, we believe that before the solution to the question like \( \text{NP} = \text{P} \), a complete theory regarding \( \text{NP} \) sets should be developed.

We start by proving that in some cases containment of two complexity classes implies separation of other complexity classes. We show in Section 2 that if \( \text{PSPACE} \) contains a superpolynomial nondeterministic time class then all nice nondeterministic time and deterministic space classes above exponential separate. This answers an open question of [HIS] who first studied this kind of question. [HIS] show that if \( \text{NP} \) contains a super-polynomial deterministic time class then all higher (nice) deterministic and nondeterministic time classes separate. However, their technique does not seem to apply in the case of time vs space. [HIS] use the fact that if \( \text{NP} \) contains a superpolynomial deterministic time class, say \( \text{DTIME}[n^\text{clogn}] \), then by diagonalization extremely sparse sets (subsets of \( \{1^2 \ldots n\} \)) can be found in \( \text{DTIME}[n^\text{clogn}] - \text{P} \), and therefore in \( \text{NP} - \text{P} \). In the case of \( \text{PSPACE} \) and \( \text{NP} \) we show that under similar assumptions very sparse sets in \( \text{NTIME}[n^\text{clogn}] \subset \text{NP} \) do not exist. Hence under the similar assumption, say \( \text{NTIME}[n^\text{clogn}] \subset \text{PSPACE} \), we must use a different technique to show the required separation.

In order to provide a complete study, in Section 2.3 we show that arbitrarily sparse sets do exist in a nontrivial \( \text{NTIME} \) hierarchy. For instance, there are arbitrarily sparse sets in \( \text{NTIME}[2^{n^\text{clogn}}] - \text{NP} \). It is important to notice that direct diagonalization is not possible here. This result is also important on its own merit.

Then, in Section 3, we study the similar question for \( \text{NTIME} \cap \text{CoNTIME} \). Incidentally, we present a very clean and simple construction of Sipser's oracle \( X \) under which \( \text{NP}^X \cap \text{CoNP}^X \) does not have complete sets. (See [S] and [HI].) Sipser's construction was quite long. Our approach is completely different and very simple.

In Section 4, we investigate the connection between \( \text{NP} \)-immunity and \( \text{NP} \)-simplicity of certain languages, and containment properties of complexity classes. A language \( L \) is called \( W \)-\text{immune} (see, for instance [BS]) if \( L \) has no infinite subset which is a member of the complexity class \( W \). A language \( L \) is called \( W \)-\text{simple} if \( L \) is in \( W \) but no infinite subset of the complement of \( L \) is in \( W \). In [B] L. Berman has shown that languages complete (with respect to polynomial time many one reductions) for exponential time are not \( P \)-immune. Berman asks whether \( \text{NP} \)-complete languages can be \( P \)-immune. Homer [H] asked whether \( \text{NP} \)-complete languages can be \( \text{NP} \)-simple. We show that if \( \text{NP} \) is exponentially hard (which is very likely), then no \( \text{NP} \) hard set can be \( \text{NP} \)-\text{immune} (in fact, it must have a dense \( \text{NP} \) subset). Also motivated by Homer's question, we show that if \( \text{NP} \cap \text{CoNP} \) is exponentially hard (which is very likely) then \( \text{NP} \)-\text{complete} \( \text{NP} \)-\text{simple} sets do not exist.

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In Section 5 we study various reducibilities to almost everywhere complex sets, a notion which is related to immunity. We generalize some results by Lynch [L].

2. Containment and Separation

One of the most important problems in computational complexity theory is to determine the relations among the natural complexity classes, e.g. $P$, $NP$, $PSPACE$, etc. Besides the classic $P=NP$ and $NP=PSPACE$ problems, there is a whole spectrum of other important problems like $E=NE=ESPACE$, $EE=NEE=EEESPACE$, etc. (Here, $E$ denotes the class of languages acceptable by a Turing machine in exponential time, $NE$ nondeterministic exponential time, $ESPACE$ exponential space, $EE$ double exponential time, $NEE$ nondeterministic double exponential time, $ESPACE$ double exponential space, and so on.) Even a solution to $P \neq NP$ or $NP \neq PSPACE$ would not necessarily settle any of the other problems. Little is known about the relations among these complexity classes. Recently, an interesting connection was established by [H1]. Define a set $S$ to be \emph{sparse} if $|S \cap S| \leq n^c + c$ for some $c$. The following was shown.

**Theorem A [H1].** There is a sparse set in $NP-P$ iff $NE \neq E$.

Related results can be found in [HIS] and [HY]. Using Theorem A, [HIS] proved the following.

**Theorem B [HIS].** Let $R(n)$ be time constructible and for all $k \geq 1$

\[
\lim_{n \to \infty} \frac{n^k}{R(n)} = 0
\]

Then $DTIME[R(n)] \subseteq NP$ implies that for any monotonically increasing, time-constructible $T(n)$, where $T(n) > 2^a$ and $2^f(n)$ is computable in time polynomial in $n$,

\[
\bigcup_{c=1}^{\infty} NTIME[T(n)^c] \neq \bigcup_{c=1}^{\infty} DTIME[T(n)^c].
\]

(Theorem B)

After seeing Theorem B, it is natural to ask whether a similar result is also true for the $NTIME$ versus $SPACE$ classes. In [HIS], this question was raised: assuming, say, $NTIME[n \log n] \subseteq PSPACE$, can we prove that $NE \neq SPACE$, $NEE \neq ESPACE$, and so on? The proof for Theorem B fails here, since we cannot apply standard diagonalization methods to nondeterministic time to construct arbitrarily sparse sets in $NTIME[n \log n]-NP$. As a matter of fact, we will construct an oracle $A$ such that $NTIME[A \log] \not\subseteq NP$. As a matter of fact, we will construct an oracle $A$ such that $NTIME[A \log] \not\subseteq NP^A$ does not contain very sparse sets. We now summarize our problems.

**Question 1.** Let $R$ be a time-constructible function such that $\frac{n^k}{R(n)} = 0$ for all $k$. Assume $NTIME[R(n)] \subseteq NP$, can we conclude that $NE \neq SPACE$, $NEE \neq ESPACE$, and so on?

**Question 2.** Let $R$ be defined as above. Do there exist arbitrarily sparse sets in $NTIME[R(n)]-NP$, or in $NTIME[n \log n]-NP$? If this is true, then Question 1 can be answered by the methods used to prove Theorem B.

**Question 3.** If Question 2 is not true, then what is the smallest class $C$ such that arbitrarily sparse sets exist in $C-NP$.

Before we present a stronger answer to Question 1, we answer Question 2. Then we answer Question 3 and some other related questions to provide a complete study. The solution of Question 3 also gives a simple solution to a classical open question concerning tally sets in the $NTIME$ hierarchy.

We need one more notation: If $C$ is a complexity class, then

\[
C|_{f(n)} = \{ S \mid S \in C \text{ and } S \subseteq \{1^f(n) \mid n=1,2, \cdots \}\}.
\]

2.1 Difference between $DTIME$ vs $NTIME$ and $NTIME$ vs $SPACE$

We first explain the proof of Theorem B by an example and then give an oracle which suggests that the same proof would not work for $NTIME$ vs $SPACE$.

**Example:** Assuming $NTIME[n \log n] \subseteq NP$ we prove $NE \neq E$ (and $NEE \neq EEE$, etc.) as follows. Let $M_1, M_2, \ldots$ be an enumeration of polynomial time machines where each machine is enumerated infinitely often. Define $L=\{1^n \mid M_1 \text{ rejects } 1^n \text{ in less than } n \log n \text{ time}\}$. Using standard techniques developed in [HLS] (see also [HU]), we can design a machine $M_L$ that on input $1^n$ in $n \log n$ time and $M_L$ accepts iff $M_1$ rejects in less than $n \log n$ steps. Obviously $L=L(M_L)$. It is not hard to see that $L \neq L(M_L)$ for all $n$, since each machine is enumerated infinitely often. Therefore $L \in DTIME[n \log n]-P$ and $L \in NP-P$ by our assumption. Then $L=\{1^n \in L \} \in NE-E$ [B2]. Note that, from the proof, the assumption of $DTIME[n \log n] \subseteq NP$ can be replaced by the weaker condition $DTIME[n \log n]|_{1^n \subseteq NP}$, or even $DTIME[R(n)]|_{1^n \subseteq NP}$ for $R(n)$ a super-polynomial function and $f(n)$ any time-constructible function.

One would naturally like to obtain a similar proof for $NTIME$ versus $SPACE$ under a similar