On Nontermination of Knuth-Bendix Algorithm

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Abstract: The well-known Knuth-Bendix completion algorithm which computes a confluent and finitely terminating term rewriting system from a given set of equations, can either terminate with success or abort or even nonterminate. Very little is known about the origin of nontermination of this algorithm. We study the structural properties of rewrite rules which cause nontermination. The notion of the crossed rules is introduced for these purposes. We look for sufficient conditions guaranteeing nontermination of algorithm in the presence of crossed rules. A special attention is devoted to a verifiable condition of such kind.

1. Introduction

The Knuth-Bendix theorem [KB 70] gives a decision procedure for the confluency of finite terminating term rewriting systems (abbr. TRS). This idea is used as a theoretical background for Knuth-Bendix completion algorithm (abbr. KBA). The aim of KBA is to construct a canonical (i.e. confluent and finite terminating) TRS from a given set of equations (or rules).

In general, KBA tackles an undecidable problem thus it is not surprising that taking a finite set of equations the procedure either halts with success, aborts or nonterminates. KBA aborts if it is not able to prove, with supplied means, the termination of the constructed TRS.

Very little is known about the origin of KBA nontermination. The following fact [De 85b] can be interpreted as a partial result in this direction. For a given theory and ordering on terms (which is used for TRS termination proof) KBA cannot both succeed and nonterminate depending on the order the equations are processed. Hence follows that the used ordering has an influence on KBA behaviour. More apparent is this fact from a little bit stronger version of the mentioned result. If for a given ordering and equations KBA nonterminates there is no (finite) canonical TRS compatible with the given ordering to the given input equations [Av 84].

This paper is an attempt to study the reasons of KBA nontermination through investigation of the structural properties of the generated rules. The first result concerning structural properties of an infinite set of rules generated by KBA is due to Huet [Hu 80b]. He revealed an interesting property of the left-hand side terms of rules which are generated in the case of KBA nontermination.

We concentrate in our approach not only on the structural properties of generated rules but mainly on a special property of rules (equations) from which an infinite set of rules could be generated. Such property is incorporated into a notion of crossed pair of rules which is in fact a special case of a critical pair.
of rules (known from the Knuth-Bendix decision procedure). A crossed pair of rules
with some additional assumptions cause KBA nontermination. The notion of crossed
pair is generalized to crossed n-tuples of rules for an arbitrary n. This
generalization is important because there are n-tuples of rules which can cause
nontermination of KBA and for each proper subset of them KBA terminates
successfully.

The sufficient conditions which guarantee nontermination of KBA in the presence
of crossed rules use a property of an infinite set of terms. A verifiable condition
has been proved for monadic TRSs only though we suppose this result can be
generalized for linear TRSs as well.

Because of lack of space the proofs has been omitted and can be found in
[HP 85] where they are presented in full detail.

2. Preliminaries

This part is a brief summary of basic notions and notations taken mostly from
[Hu 80b] and [De 85b].

The set of terms constructed from a set of function symbols F and variables X
is denoted T(F,X). Terms are usually denoted by letters t,s,u,v..., variables by x,
y, z. A set of occurrences D(t) of a term t is used for unique identification of all
subterms of t. Elements of D(t) are strings over N and are usually denoted as a,b.
A subterm of term t identified by occurrence a ∈ D(t) is denoted t[a]. The result of
replacing a subterm t/a in the term t by a term v is denoted t[a[v]. Two terms
t_1, t_2 are unifiable if there are substitutions (mappings from X to T(F,X))
such that \( σ_1 t_1 = σ_2 t_2 \). For a substitution σ the term σt is an instance of t.

A term rewriting system R is a finite set of rewrite rules s → t over the
symbols F. A term u is R-reducible if there are a rule s → R, a substitution \( σ \) and
a ∈ D(u) such that \( σs = u/a \) (if not, the term u is R-irreducible). This
R-reducible term u can be rewritten to \( v = u[a←R] \) and such rewriting step is
denoted \( u → v \). R-derivation is a sequence of rewriting steps \( u_1 → u_2 → \ldots \). The
notation \( u →^* v \) indicates a derivation sequence \( u → \ldots → v \). A term \( t = R(u) \) is a
R-normal form of a term u iff \( u →^* t \) and t is R-irreducible. Denote
\( \text{depth}(R) = \max\{\text{depth}(s → t) \mid s → t ∈ R\} \) as depth
of a rule, resp. a set of rules.

A TRS R is (finitely) terminating if there is no infinite R-derivation and
confluent if for all terms s, u, v such that \( s →^* u \) and \( s →^* v \) there is a term t
such that \( u →^* t \) and \( v →^* t \) holds. TRS R is canonical if it is terminating and
confluent. Both properties, confluency and termination, are undecidable.

Two rules \( s_1 → t_1, s_2 → t_2 \) form a critical pair of rules if \( s_2 \) overlaps \( s_1 \), i.e.
there are substitutions \( σ_1, σ_2 \) and a ∈ D(s_1) such that \( σ_1 s_1/a = σ_2 s_2 \). Then a
critical pair of terms \( (σ_1 s_1[a ← σ_2 t_2], σ_1 t_1) \) represents two different ways to rewrite