Gröbner bases and Buchberger's algorithm to compute them (BUC1,2,3) and the related concept of standard bases (HIR) have become a very useful tool for constructive problems in polynomial ideal theory and related domains (e.g. solving systems of algebraic equations).

The aim of this paper is to propose a generalization of these techniques to non-commutative polynomial rings.

The problem arising in the generalization stems from the fact that a non-commutative polynomial ring is not noetherian: in particular Dickson's Lemma, which plays a crucial role both in Buchberger's and in the standard basis algorithm, does not hold in a non-commutative polynomial ring.

This has no effect on the concept of Gröbner basis since (with the obvious modifications required by non-commutativity) the main characterizations of Gröbner bases still hold in the non-commutative case.

It has however a strong effect on the constructive aspect of the problem, since, because of non-noetherianity, a finitely generated ideal may have no finite Gröbner basis, at least w.r.t. a given term ordering. We are able, however, to present a semi-decision procedure that computes a finite Gröbner basis for a finitely generated ideal w.r.t. some term ordering, in case a finite Gröbner basis exists.

Different generalizations of Buchberger's algorithm for the non-commutative case that retain Dickson's Lemma have been provided recently (A-L,GAL).

The effect of non-noetherianity is more significant if we consider standard bases: the usual way of introducing them as "good" bases either in a completion or in a localization of the polynomial ring cannot be straightforwardly generalized to the non-commutative case. We will, however, delay a discussion of this problem to a different publication.

While the first part of the paper will be sufficiently general to allow for a common treatment of Gröbner and standard bases, we will then restrict our interest to Gröbner bases and to semidecision procedures to compute them.
NOTATIONS

If $B$ is a subset of a ring $R$, we will denote by $B^*$ the set \( \{ b \in B : b \neq 0 \} \).

1. DEFINITIONS

1.1 Let $S$ denote a free semigroup generated by a finite alphabet $A$.

If $m,n$ are in $S$, we will say $m$ is a multiple of $n$ ( $n$ divides $m$ ) iff there are $l,r$ in $S$ s.t. $m = lr$.

$K[S]$, $K$ a field, will denote the ring whose elements are finite linear combinations
of elements of $S$, $K[S] := \{ \sum_{i=1}^{n} c_i m_i : c_i \in K^*, m_i \in S \}$, with multiplication canonically defined in terms of the semigroup multiplication.

1.2 A term ordering on $S$ is a total ordering $\prec$ s.t.:

i) for all $m,m_1,m_2$ in $S$, $m_1 \prec m_2$ implies $mm_1 \prec mm_2$ and $m_1m \prec m_2m$.

ii) for all $m$ in $S$, there exists no infinite decreasing sequence $m_1 \succ ... \succ m \succ ...$ s.t. for all $i$ $m_i > m$.

A term ordering will be called positive iff $1 \prec m$ for all $m$ in $S$, or, equivalently, iff $m \prec mn$ and $m \prec nm$ for all $m,n$ in $S$.

A term ordering will be called negative iff $1 \succ m$ for all $m$ in $S$.

1.3 Let $\prec$ be a term ordering on $S$. If $f := \prod_{i=1}^{n} c_i m_i$, $m_i$ in $S$, $m_1 > m_2 > ... > m_t$, define $M_T(f) := m_1$, $lc(f) := c_1$.

If $G \subseteq K[S]$, define $M_T(G) := \{ M_T(f) : f \in G^* \}$.

If $I$ is a two-sided non-zero ideal of $K[S]$, $M_T(I)$ is a two-sided ideal of $S$.

1.4 A distinguished set (shortly a d-set) for $I$ is a set $F \subseteq I^*$ s.t. $M_T(F)$ generates $M_T(I)$.

1.5 Definitions and notations given above are obvious generalizations to the non-commutative case of those related, in the commutative case, to Gröbner and standard bases, so that, in the commutative case, Gröbner (resp. standard) bases are d-sets for positive (resp. negative) term orderings (BUCL,2,HIR,LAZ,MM,ROB).

1.6 It is to be reminded that, unlike in the commutative case, Dickson's Lemma (which is crucial in many arguments) does not hold any more.

Dickson's Lemma states that, for every infinite sequence of (commutative) terms,
$m_1, ..., m_i, ...$, there is an index $N$ s.t., if $i > N$, there exists $j \leq N$ s.t. $m_i$ is a multiple of $m_j$.

A counterexample to Dickson's Lemma in the non-commutative case is for instance the