COMPARING CATEGORIES OF DOMAINS

by

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Abstract. We discuss some of the reasons for the proliferation of categories of domains suggested for the mathematical foundations of the Scott-Strachey theory of programming semantics. Five general conditions are presented which such a category should satisfy and they are used to motivate a number of examples. An attempt is made to survey some of the methods whereby these examples may be compared and their relationships expressed. We also ask a few mathematical questions about the examples.

1. Introduction.

A great variety of mathematical structures have been proposed for use as semantic domains for programming languages. We focus on one line of investigation which uses certain classes of partially ordered sets and aims to give a semantics which is denotational in nature. This approach was introduced by Dana Scott and Chris Strachey in the late sixties ([24], [30]) and it remains an area of active research today. The original category used by Scott and Strachey had complete lattices as objects and monotone functions that preserve least upper bounds of directed collections as arrows. But in the decade and a half since their work a host of other closely related categories have been investigated. Discussing the reasons that these alternatives have been suggested and the relationships between the different categories is the goal of the current document. A secondary objective is to ask a few mathematical questions about the categories. Most of the questions mentioned are not motivated by any particular problem in programming semantics. It is hoped, however, that they will evoke the curiosity of the reader as they have that of the author.

The paper is divided into four sections and an appendix. Section two discusses some of the conditions from programming semantics which motivate the choice of a category of domains. A collection of five such conditions are enumerated and we discuss how these conditions are satisfied to one degree or another by specific categories. Section three discusses what might be called "distinguishing conditions" on categories. The most important of these is Smyth's Theorem and we state some of its generalizations. The fourth section introduces the categories of "continuous

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domains" which are a current area of investigation. Proofs of most of the theorems stated below can be found scattered throughout the literature (see, in particular, [4], [17] and [29]). A few short proofs that do not require much background have been included in an appendix. A result whose proof may be found there is marked with an asterisk (*).

The reader is assumed to be familiar with the following concepts from category theory: category, functor, object, arrow, product and coproduct, terminal and initial objects, equivalence and isomorphism between categories, inverse limit, and continuous functor. Definitions may be found in any of the standard references on category theory ([1], [5], [13]).

2. In search of the perfect category of domains.

Basic definitions and notation. A poset is a set with a binary relation $\leq$ which is reflexive, antisymmetric and transitive. If $D$ is a poset, a subset $M \subseteq D$ is directed if every finite subset of $M$ has a bound in $M$. A poset $D$ is directed complete if every directed $M \subseteq D$ has a least upper bound $\bigcup M$ in $D$. A function $f : D \to E$ is Scott continuous if it is monotone and $f(\bigcup M) = \bigcup f(M)$ for each directed $M \subseteq D$. The term "continuous" comes form the fact that it is possible to define a topology on a dcpo which makes these directed lub preserving maps exactly the continuous functions. If $D$ is a dcpo then a subset $\mathcal{O} \subseteq D$ is said to be Scott open if

1. If $x \in \mathcal{O}$ and $x \leq y$ then $y \in \mathcal{O}$.
2. If $M$ is directed and $\bigcup M \in \mathcal{O}$ then $M \cap \mathcal{O} \neq \emptyset$.

These open sets form a $T_0$ topology on $D$ called the Scott topology.

We denote by DCPO the category of directed complete posets and continuous functions. All of the domains that we consider below will be dcpo's. We therefore adopt the following convention: unless otherwise stated, every category $C$ is assumed to be a full subcategory of DCPO. Here are a few examples:

- Let $S$ be any set. Order $S$ discretely, i.e. $x \leq y$ iff $x = y$. These discretely ordered posets are all dcpo's. If $S$ and $T$ are discretely ordered posets then any function $f : S \to T$ is continuous.
- Any finite poset is a dcpo.
- If $\alpha$ is an ordinal then $\alpha + 1$ is a dcpo.
- If $S$ is a set then the powerset of $S$, ordered by set inclusion, is a dcpo.
- The extended reals (i.e. the reals under the usual ordering with a largest and smallest element added) form a dcpo.

DCPO is a rather "large" category. Indeed, the discretely ordered posets mentioned above form a full subcategory $\text{Set} \subseteq \text{DCPO}$ which is isomorphic to the category of sets. There are, however, familiar partially ordered sets which are not dcpo's. For example, the rational numbers do not