1. INTRODUCTION A long-standing problem in domain theory has been the search for algebra structures that ride naturally on the ordered sets involved. Indeed, the constructions in the theory of complete partial orders and continuous lattices, as applied to the recursive definition of a data structure, are usually independent of any algebra carried by the data structure, and do not mesh nicely with the algebra. The aim of the current paper is to provide an introduction to the kind of algebra that does guarantee a good mesh with partial orderings, and to present topological ideas and category-theoretical relationships showing how the algebra is automatically reproduced under order-theoretical constructions such as power domains. As a potential application of the algebra, a new direct approach to the programming of geometry and scientific models is proposed.

Giving a brief survey of an extensive theory, this paper is necessarily somewhat condensed. Fuller details of many of the topics treated here, as well as an introduction to the universal algebraic notations used, may be found in [6]. Note, too, that the whole theory as described here has so far only been worked out on the basis of finitary universal algebraic methods. There is great potential for future development of the theory using the monadic approach to algebra. Another aspect of the theory that has hardly been investigated at all is that of duality. Here, too, a great deal remains to be done.

2. MODES The basic algebraic concept is that of a mode. A mode is an algebra \((A, \Omega)\) satisfying the following conditions:

(2.1) The algebra is idempotent, i.e. each singleton subset \(\{a\}\) of \(A\) is a subalgebra \((\{a\}, \Omega)\) of \((A, \Omega)\); and
the algebra is entropic, i.e. each operation \( \omega \) in \( \Omega \) (of arity \( \omega_T \)),

already a set mapping \( \omega : A_1^{\omega_T} \rightarrow A; (x_1, \ldots, x_{\omega_T}) \mapsto x_1 \cdots x_{\omega_T} \), is also a

homomorphism \( \omega : (A_1^{\omega_T}, \Omega) \rightarrow (A, \Omega) \).

Some examples will serve to demonstrate the scope of this apparently restrictive
definition, showing how various familiar mathematical concepts are brought into the
purview of modal theory.

**EXAMPLE 2.3.** If \( \Omega \) is empty, the conditions (2.1) and (2.2) are vacuously
satisfied. Thus unstructured sets \( A \) are modes.

**EXAMPLE 2.4.** A semilattice \( (L, \cdot) \) is a mode. Idempotence reduces to \( x \cdot x = x \)
for \( x \) in \( L \), while the entropic law \( (x \cdot y) \cdot (z \cdot t) = (x \cdot z) \cdot (y \cdot t) \) for the single
infix binary operation \( \cdot \) follows from the commutative and associative laws. The
semilattice \( (L, \cdot) \) has a partial order specified by

\[
(2.5) \quad x \preceq y \quad \text{if and only if} \quad x \cdot y = x.
\]

With this order, \( (L, \cdot) \) is called a *meet semilattice*. If a partial order \( \preceq \) is
given on a set \( L \), and this partial order is known to come from a meet semilattice
structure on \( L \) (i.e. each pair of elements of \( L \) has a greatest lower bound),
then (2.5) may be read backwards to specify the semilattice operation \( \cdot \) on \( L \).
Sometimes the dual notion of *join semilattice* \( (L, +) \) is used: \( x \preceq y \) if and only
if \( x + y = y \). Meet and join semilattices are the means by which order is dealt
with in modal theory.

**EXAMPLE 2.6.** Let \( E \) be a vector space over a field \( R \), or more generally a unital
module over a commutative ring \( R \) with 1. For each \( r \) in \( R \), define a binary
operation \( r \) by

\[
(2.7) \quad r : E \times E \rightarrow E; (x, y) \mapsto x(1-r) + yr.
\]

Interpreting \( R \) as the set of these binary operations \( r \), the algebra
\( (E, R) \) becomes a mode. Idempotence follows since \( xxr = x(1-r) + xr = xl = x \), and a