COVERING MINIMA AND LATTICE POINT FREE
CONVEX BODIES

by
Ravi Kannan*  László Lovász**

Abstract

Suppose $K$ is a convex set of nonzero volume in Euclidean $n$–space $\mathbb{R}^n$ and it is symmetric about the origin (i.e., if $x$ belongs to $K$, so does $-x$). For any real number $t$, let $tK = \{tx : x \in K\}$. The infimum over all positive real numbers $t$ such that if a copy of $tK$ is placed centered at every integer point, all of $\mathbb{R}^n$ is covered, is called the “covering radius” of $K$ (with respect to the lattice $\mathbb{Z}^n$). The covering radius and related quantities have been studied extensively in Geometry of Numbers. In this paper, we define and study the “covering minima” of a convex body which is not necessarily symmetric about the origin; the covering radius will be a special case of one of these minima. This extension to general convex bodies has among other things, applications to algorithms for Integer Programming which was our initial motivation. This motivation is explained in some detail later. We use the results of the paper to derive bounds on the width of lattice point free convex bodies and analyze their structure.

Notation

$\mathbb{R}^n, \mathbb{Z}^n$ denote the set of $n$–vectors with real, respectively integer components. For any two vectors $v, u$ in $\mathbb{R}^n$, $(v, u)$ denotes the scaler product of the two vectors. For any two sets $S, T$ in $\mathbb{R}^n$, $S + T$ denotes the set $\{s + t : s \in S, t \in T\}$. Thus for example, for a convex body $K$, which is symmetric about the origin, $K + \mathbb{Z}^n$, is the union of copies of $K$ placed with each integer point as a centre. Thus, the covering radius mentioned above is the infimum over all positive reals $t$ such that $tK + \mathbb{Z}^n$ contains all of $\mathbb{R}^n$. We abbreviate $S + T$ by $S + t$ if $T$ is the singleton set $\{t\}.$

If $V$ is a subspace of $\mathbb{R}^n$ and $V^\perp$ the orthogonal complement subspace, then for any set $S$ in $\mathbb{R}^n$, we define the projection of $S$ parallel to $V$ denoted $S/V$ to be the set $\{t : t \in V^\perp; \exists s \in V$ such that $s + t \in S\}$. $S/V$ may be pictured as the projection of $S$ “onto” $V^\perp$. For $c \in \mathbb{R}^n, r \in \mathbb{R}, B(c, r)$ is the closed ball of radius $r$ with $c$ as centre. For any set $S$, int$(S)$ denotes the interior of the set $S$.

* Department of Computer Science, Carnegie-Mellon University, Pittsburgh ; supported by NSF Grant ECS-8418392
** Institute of Mathematics, Eötvös Loránd University, Budapest
If \( K \) is a 0-symmetric convex body in \( \mathbb{R}^n \) and \( L \) a lattice in \( \mathbb{R}^n \), the successive minima (Cassels 1971) of \( L \) with respect to \( K \) are denoted \( \lambda_1(K, L), \lambda_2(K, L), \ldots \lambda_n(K, L) \).

The name \( c_0 \) is reserved for the numerical constant occuring in lemma (1.4) which is used in several places.

**Section 1 : Introduction**

Let \( K \) be a convex body (a convex set of nonzero volume) in \( \mathbb{R}^n \). We define the \( j \) th covering minimum \( \mu_j \) of \( K \) to be the infimum of all positive reals \( t \) such that \( tK + \mathbb{Z}^n \) intersects all \( n - j \) dimensional affine subspaces of \( \mathbb{R}^n \). In the case that the body \( K \) is symmetric about the origin, its \( \mu_n \) clearly equals the covering radius. It is also clear that

\[
0 = \mu_0 \leq \mu_1 \leq \mu_2 \ldots \leq \mu_n.
\]

The quantities \( \mu_j \) are invariant under translations of \( K \), since a translation maps the set of \( n - j \) dimensional affine subspaces onto itself. Our main results on the covering minima is that \( \mu_j \leq c_0 j^2 \mu_1 \) for all convex bodies \( K \) where \( c_0 \) is a numerical constant independent of \( n, K \). In words, if \( K + \mathbb{Z}^n \) already intersects every hyperplane, then a dilation of \( K \) by a factor of \( c_0 j^2 \) meets every \( n - j \) dimensional affine space. It is simple to show that \( \mu_1 \) equals the reciprocal of the "lattice width" of \( K \) which is defined below. This combined with our inequality \( \mu_j \leq c_0 j^2 \mu_1 \), yields a bound of \( c_0 n^2 \) on the lattice width of any convex body containing no lattice points. Since this was our initial motivation for the work, we explain it in some detail here.

The "lattice width" of a convex set \( K \) along a vector \( v \) is is the quantity \[
\max \{(v, x) : x \in K\} - \min \{(v, x) : x \in K\}.
\]

The "lattice width" of \( K \) is the minimum over all integer vectors \( v \) of the lattice width of \( K \) along \( v \). If the lattice width of \( K \) along \( v \), is \( \alpha \), then there are at most \( \alpha + 1 \) hyperplanes of the form \( \{x : (v, x) = z\} \), \( z \) an integer that intersect \( K \). H.W.Lenstra (1981, 1983) used this idea to reduce an \( n \)-variable integer program to a bounded number of \( n - 1 \) dimensional integer programs to derive his polynomial time algorithm for integer programs in a fixed number of dimensions.

Integer Programming (feasibility question) is the problem of determining whether a given polytope \( P \) contains an integer point. Lenstra gives a polynomial time algorithm that for any polytope \( P \) in \( \mathbb{R}^n \) either finds an integer point in \( P \) or finds an integer vector \( v \) so that the lattice width of \( P \) along \( v \) is less than \( c n^2 \) where \( c \) is a constant independent of \( n \). Every integer point must lie on a hyperplane of the form \( (v, x) = z \) for some integer \( z \), and there are at most \( c n^2 + 1 \) such hyperplanes intersecting \( P \). It obviously suffices to determine for each such hyperplane \( H \), whether \( H \cap P \) contains an integer point. Lenstra uses this to show that an \( n \) variable problem can be reduced to \( c n^2 \) problems each in