9. Parallel machines problems

9.1 Problems with identical parallel machines

9.1.1 The $P2|pmtn, d_i|\epsilon(L_{max}/C_{max})$ problem

[Mohri et al., 1999] are interested in a bicriteria scheduling problem where two machines are available to process n independent jobs that can be preempted at any (real) time. Each job J_i is defined by a processing time p_i and a due date d_i. Without loss of generality we assume that $d_1 \leq d_2 \leq ... \leq d_n$. The aim is to schedule the jobs in such a way that the makespan C_{max} and the maximum lateness L_{max} are minimised. By considering the ϵ constraint approach they provide a characterisation of strictly non dominated criteria vectors. This problem is solvable in polynomial time.

Firstly, Mohri, Masuda and Ishii tackle the $P2|pmtn, d_i|L_{max}$ problem which can be solved by iteratively solving $P2|pmtn, \widetilde{d_i}|-$ problems using the procedure of [Sahni, 1979]. At each iteration, a $P2|pmtn, \widetilde{d_i} = d_i + L|-$ problem is solved: if a feasible solution to this problem exists, then a schedule for which the value of the maximum lateness criterion is equal to L exists. Otherwise, no such schedule exists. Starting with Sahni's procedure they show the following result.

Lemma 37 *[Mohri et al., 1999]*
The optimal value of the L_{max} criterion for the $P2|pmtn, d_i|L_{max}$ problem is given by:

$$L^*_{max} = \frac{1}{2} \max_{i=1,...,n} \Big(\sum_{j=1}^{i} p_j - \min_{k=1,...,i} (d_k + \sum_{j=k+1}^{i} p_j) - \min_{k=1,...,i-1} (d_k + \sum_{j=k+1}^{i-1} p_j)\Big),$$

under the assumption that $p_i \leq d_i$, $\forall i = 1, ..., n$.

This result is extended to the $P2|pmtn, d_i|\epsilon(L_{max}/C_{max})$ problem when the constraint on the makespan is fixed, *i.e.* when we have $C_{max} \leq \epsilon$. In this case we have the following result.

Lemma 38 *[Mohri et al., 1999]*
The optimal value of the L_{max} criterion for the $P2|pmtn, d_i, C_{max} \leq \epsilon|L_{max}$ problem is denoted by L^ϵ_{max}. We have:

$$L^\epsilon_{max} = \max\{L^*_{max}; \max_{i=1,\ldots,n} \Big(\sum_{j=1}^{i} p_j - \epsilon - \min_{k=1,\ldots,i-1} (d_k + \sum_{j=k+1}^{i-1} p_j)\Big)\},$$

under the assumption that $p_i \leq d_i$, $\forall i = 1, \ldots, n$.

The resolution of the $P2|pmtn, d_i|\epsilon(L_{max}/C_{max})$ problem with a fixed value ϵ is similar to the resolution of the $P2|pmtn, d_i|L_{max}$ problem. The only difference lies in the construction of the deadlines at each iteration. For the bicriteria problem we have $\widetilde{d_i} = \min(d_i + L; \epsilon)$, $\forall i = 1, \ldots, n$. Therefore, using Sahni's procedure, if we found a feasible schedule for these deadlines then a schedule exists for which the makespan is lower than ϵ and the value of the maximum lateness is equal to L.

Mohri, Masuda and Ishii propose a characterisation of the set E. They identify a sufficient condition for the existence of a single strictly non dominated criteria vector. This condition can be seen as a consequence of the mathematical expression of L^ϵ_{max}.

Theorem 31 *[Mohri et al., 1999]*
Let $F = \max_{i=1,\ldots,n} \Big(\sum_{j=1}^{i} p_j - \min_{k=1,\ldots,i-1} (d_k + \sum_{j=k+1}^{i-1} p_j)\Big)$. If $L^*_{max} \geq F - C^*_{max}$ then there exists a single strictly non dominated criteria vector defined by $[C^*_{max}; L^*_{max}]^T$. C^*_{max} is the optimal value of the makespan for the $P2|pmtn|C_{max}$ problem can be stated as follows ([McNaughton, 1959]):

$$C^*_{max} = \max(\max_{i=1,\ldots,n} p_i; \frac{1}{2} \sum_{i=1}^{n} p_i).$$

If the condition of theorem 31 does not hold the set E is contained, in the criteria space, in a line segment limited by the criteria vectors $[C^*_{max}; F - C^*_{max}]^T$ and $[C^+_{max}; L^*_{max}]^T$, where C^+_{max} is the minimal value of the makespan of an optimal schedule for the L_{max} criterion. The value C^+_{max} can be obtained from the feasible schedule returned by Sahni's procedure for the $P2|pmtn, \widetilde{d_i} = d_i + L^*_{max}|-$ problem. Besides, for any given criteria vector $[C; L]^T$ on the line segment a corresponding schedule is obtained using Sahni's procedure with $\widetilde{d_i} = \min(d_i + L; C)$, $\forall i = 1, \ldots, n$.

Example.
We consider a problem for which $n = 5$.

i	1	2	3	4	5
p_i	3	7	4	8	10
d_i	5	10	12	13	15