4. A General Procedure for Stress Integration and Applications in Metal Plasticity

We emphasized in Chapter 2 that a central point in any inelastic finite element analysis is the calculation of the stresses in an effective manner. The stress calculation predicts the material response described by the material model. In this chapter we present effective numerical procedures for the stress calculation in metal plasticity.

We first discuss in Section 4.1 the role of stress calculation in a finite element incremental-iterative analysis. Then, in Section 4.2 we establish a general approach for the implicit stress computation within the time (load) step, and apply the approach to a class of time independent plasticity models. We call this approach the *governing parameter method* because the stress integration problem is reduced to solving one nonlinear equation for a governing parameter. The governing parameter method is applicable to models for which the mean stress does not affect the inelastic behavior (isochoric inelastic deformation), as in metal plasticity, viscoplasticity and/or creep models, as well as to models with volumetric inelastic deformation, as in geological plasticity models. In this and in the subsequent chapters the algorithm will be presented for a number of commonly used material models.

In Section 4.3 we give a brief review of stress integration algorithms as they relate to the governing parameter method. A detailed derivation of the relations used in the governing parameter method for the stress integration of the von Mises model is presented in Section 4.4, and Section 4.5 contains a number of solved examples. Finally, in Section 4.6 we derive the computational procedure for Hill's othotropic model and give example solutions.

4.1 Introduction

In any structural analysis we need to address how to represent and model the deformations and the material behavior. For the representation of the deformations an adequate displacement field must be assumed, and the corresponding kinematic quantities, such as the strains, strain rates, deformation gradient, etc. need be calculated. This was briefly discussed in Chapter 2, while a deeper presentation, using the same notation, is given in Bathe (1996).

The material behavior is described by material mathematical models and we consider here phenomenological models based on experimental observations. In order to use a material model in an engineering analysis, it is necessary to have numerical procedures that provide an accurate calculation of the model response, according to the selected model parameters and the loading conditions. In this book we present numerical algorithms for the stress integration for *strain-driven problem formulations*; these problems arise in the displacement-based and mixed finite element formulations. We consider inelastic material deformations, and our task is to calculate the stresses and inelastic strains at the end of the time (load) step, with a known stress/strain state at the start of the time step and for given strain increments. By this calculation (stress integration) we trace the history of the material deformation at a material point, and in the whole structure, under given incremental loading conditions.

The typical calculations performed in an incremental (finite element) analysis, with nonlinear material behavior, are summarized in Table 4.1.1 (see also Bathe 1996). We consider materially-nonlinear-only (*MNO*) problems (see Chapter 2).

The variables appearing in the table are the total and inelastic strains $\mathbf{e}$ and $\mathbf{e}^{IN}$, the stresses $\boldsymbol{\sigma}$, the internal variables of the material model $\boldsymbol{\beta}$, the linear strain-displacement transformation matrix $\mathbf{B}_L$ (see Section 2.3), the integration weight due to numerical integration W, the numerical integration point associated volume ΔV, and the vector of external loads $\mathbf{R}$. As in Chapter 2, the left superscripts "t" and "$t + \Delta t$" denote the start and end of time step Δt, and the right superscript denotes the iteration number. Note that for the iteration counter $i = 1$, the quantities with the upper right index equal to zero have the values corresponding to the start of the time step, e.g., ${}^{t+\Delta t}\mathbf{e}^{(0)} = {}^{t}\mathbf{e}$, ${}^{t+\Delta t}\boldsymbol{\sigma}^{(0)} = {}^{t}\boldsymbol{\sigma}$.

Referring to the table, we see that the stress integration and calculation of the tangent constitutive relations represent key steps in an inelastic incremental analysis. These calculations must be performed in a *robust, accurate, and efficient* manner. We define an algorithm as robust if it provides solutions for stresses under any reasonable boundary and loading conditions, and for relatively large strain (or load) increments. It is very important in general engineering analysis that the algorithm does not have limitations in its range of applicability, and that it does not contain any numerical instability. Of course, the algorithm should in addition yield accurate and efficient solutions. As will be seen, our primary goal is to meet these requirements.

The solution of nonlinear structural problems is obtained by incrementing the loads (or strains) in each time step Δt. Of course, in general, there is some numerical error in the stress calculation as a consequence of the numerical approximations used in the stress evaluation. The algorithm should provide reasonable accuracy for relatively large increments in strains, and the error should rapidly diminish as the strain increments are decreased. In Section