Models of Instantaneous Forward Rates

The Heath, Jarrow and Morton approach to term structure modelling is based on an exogenous specification of the dynamics of instantaneous, continuously compounded forward rates $f(t, T)$. For any fixed maturity $T \leq T^*$, the dynamics of the forward rate $f(t, T)$ are (cf. Heath et al. (1990a, 1992a))

$$df(t, T) = \alpha(t, T)dt + \sigma(t, T) \cdot dW_t,$$  \hspace{1cm} (11.1)

where $\alpha$ and $\sigma$ are adapted stochastic processes with values in $\mathbb{R}$ and $\mathbb{R}^d$ respectively, and $W$ is a $d$-dimensional standard Brownian motion with respect to the underlying probability measure $\mathbb{P}$ (to be interpreted as the actual probability). For any fixed maturity date $T$, the initial condition $f(0, T)$ is determined by the current value of the continuously compounded forward rate for the future date $T$ that prevails at time 0. The price $B(t, T)$ of a zero-coupon bond maturing at the date $T \leq T^*$ can be recovered from the formula (cf. (9.4))

$$B(t, T) = \exp \left( - \int_t^T f(t, u) \, du \right), \quad \forall t \in [0, T],$$  \hspace{1cm} (11.2)

provided that the integral on the right-hand side of (11.2) exists (with probability 1). Leaving the technical assumptions aside, the first question that should be addressed is the absence of arbitrage in a financial market model involving all bonds with differing maturities as primary traded securities. As expected, the answer to this question can be formulated in terms of the existence of a suitably defined martingale measure. It appears that in an arbitrage-free setting – that is, under the martingale probability – the drift coefficient $\alpha$ in the dynamics (11.1) of the forward rate is uniquely determined by the volatility coefficient $\sigma$, and a stochastic process that can be interpreted as the risk premium. More importantly, if $\sigma$ follows a deterministic function then the valuation results for interest rate-sensitive derivatives appear to be independent of the choice of the risk premium. In this sense, the choice of a particular model from the broad class of Heath-Jarrow-Morton (HJM) models hinges uniquely on the specification of the volatility coefficient $\sigma$. For this specific feature of continuous-time forward rate modelling to hold, we need to restrict our attention to the class of HJM models with deterministic coefficient $\sigma$; that is, to Gaussian HJM models.
In this chapter, the forward measure methodology is employed in arbitrage pricing of interest rate derivative securities in a Gaussian HJM framework. By a Gaussian HJM framework we mean in fact any model of the term structure, either based on the short-term rate or on forward rates, in which all bond price volatilities (as well as the volatility of any other underlying asset) follow deterministic functions. This assumption is made for expositional simplicity; it is not a necessary condition in order to obtain a closed-form solution for the price of a particular option, however. For instance, when a European option on a specific asset is examined in order to obtain an explicit expression for its arbitrage price, it is in fact enough to assume that the volatility of the forward price of the underlying asset for the settlement date coinciding with the option’s maturity date is deterministic.

This chapter is organized as follows. In the first two sections, we examine the general HJM set-up and the Gaussian HJM set-up respectively. Sect. 11.3 deals with issues related to the valuation of European options on stocks, zero-coupon bonds and coupon-bearing bonds. As was indicated already, we postulate that the bond price volatilities, as well as the volatility of the option’s underlying asset, follow deterministic functions. The next section is devoted to the study of futures prices and to arbitrage valuation of futures options. We focus on a straightforward derivation of partial differential equations associated with the arbitrage price of spot and futures contingent claims in a framework of stochastic interest rates. The fundamental valuation formulas for European options, established previously by means of the forward measure approach, are re-derived by solving the corresponding terminal value problems. It is worth observing that the standard Black-Scholes PDE can be seen as a special case of the PDEs obtained in Sect. 11.6.

Let us note that an efficient valuation of American-style options under uncertainty of interest rates is a rather difficult problem, and relatively little is known on this topic. It appears that the rational exercise policy of an American bond option or an American swaption cannot, in general, be described in terms of the short-term interest rate. For more details on this issue, we refer to Tanudjaja (1996), who examined various approximation techniques associated with the pricing of American-style options in a Gaussian HJM framework.

### 11.1 Heath-Jarrow-Morton Methodology

Let us first present briefly a discrete-time predecessor of the HJM model, introduced by Ho and Lee (1986). Basically, the idea is to model the uncertain behavior of the yield curve as a whole, as opposed to modelling the short-term rate representing a single point on this curve (specifically, its short side end). In view of its discrete-time feature, the Ho and Lee model of the forward rate dynamics can also be seen as a distant relative of the CRR binomial model. In contrast to the real-valued CRR process, Ho and Lee take the class of all continuous functions on \( \mathbb{R}_+ \) as the state-space for their model; any such function represents a particular shape of the yield

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1 Putting aside some particular cases when the short-term interest rate follows a specific diffusion process (cf. Sect. 10.1.4).