Continuous-time Security Markets

This chapter furnishes a summary of basic results associated with continuous-time financial modelling. The first section deals with a continuous-time model, which is based on the Itô stochastic integral with respect to a semimartingale. Such a model of financial market, in which the arbitrage-free property hinges on the chosen class of admissible trading strategies, is termed the standard market model hereafter. We discuss the relevance of a judicious choice of a numeraire asset. On a more theoretical side, we briefly comment on the class of results – informally referred to as a fundamental theorem of asset pricing – which say, roughly, that the absence of arbitrage opportunities is equivalent to the existence of a martingale measure. The theory developed in this chapter applies both to stock markets and bond markets. It can thus be seen as a theoretical background to the second part of this text.

For simplicity, we restrict ourselves to the case of processes with continuous sample paths. Putting aside a somewhat higher level of technical complexity, jump-diffusion or Lévy-type models of discontinuous prices can be dealt with along the same lines. As was mentioned already, in a typical jump-diffusion model, price discontinuities are introduced through a Poisson component. In this regard, we refer to Cox and Ross (1975), Aase (1988), Madan et al. (1989), Elliott and Kopp (1990), Naik and Lee (1990), Shirakawa (1991), Ahn (1992), Mercurio and Runggaldier (1993), Cutland et al. (1993a), Björk (1995), Mulinacci (1996) or Scott (1997).

The second section deals with a particular example of a market model – the multidimensional Black-Scholes market. In contrast to Chap. 3 and 6, we focus on general questions such as market completeness, rather than on explicit valuation of contingent claims. Since the pricing of particular claims such as options is not examined in detail, let us mention here that in a complete multidimensional Black-Scholes model with constant interest rate and stock price volatility matrix, it is straightforward to derive a PDE – analogous to the Black-Scholes PDE – which is satisfied by the price of any path-independent European claim.
8.1 Standard Market Models

Consider a continuous-time economy with a trading interval \([0, T^*]\) for a fixed horizon date \(T^* > 0\). Uncertainty in the economy is modelled by means of a family of complete filtered probability spaces \((\Omega, \mathcal{F}, \mathbb{P}), \mathbb{P} \in \mathcal{P}\), where \(\mathcal{P}\) is a collection of mutually equivalent probability measures on \((\Omega, \mathcal{F}_{T^*})\). Each individual in the economy is characterized by a subjective probability measure \(\mathbb{P}\) from \(\mathcal{P}\). Events in our economy are revealed over time – simultaneously to all individuals – according to the filtration \(\mathcal{F} = (\mathcal{F}_t)_{t \in [0, T^*]}\), which is assumed to satisfy the “usual conditions”, meaning that (a) the underlying filtration \(\mathcal{F}\) is right-continuous, i.e., \(\mathcal{F}_t = \bigcap_{u>t} \mathcal{F}_u\) for every \(t < T^*\); (b) \(\mathcal{F}_0\) contains all null sets, i.e., if \(B \subset A \in \mathcal{F}_0\) and \(\mathbb{P}\{A\} = 0\), then \(B \in \mathcal{F}_0\). We find it convenient to assume that the \(\sigma\)-field \(\mathcal{F}_0\) is \(\mathbb{P}\)-trivial (for some, and thus for all, \(\mathbb{P} \in \mathcal{P}\)); that is, for every \(A \in \mathcal{F}_0\) either \(\mathbb{P}\{A\} = 0\) or \(\mathbb{P}\{A\} = 1\).

There are \(k\) primary traded securities whose price processes are given by stochastic processes \(Z^1, \ldots, Z^k\). We assume that \(Z = (Z^1, \ldots, Z^k)\) follows a continuous, \(\mathbb{R}^k\)-valued semimartingale on \((\Omega, \mathcal{F}, \mathbb{P})\) for some – and thus for all – \(\mathbb{P} \in \mathcal{P}\). This means that each process \(Z^i\) admits a unique decomposition \(Z^i = Z^i_0 + M^i + A^i\), where \(M^i\) is a continuous local martingale, and \(A^i\) is a continuous, adapted process of finite variation, with \(M^i_0 = A^i_0 = 0\). For the definition and properties of the vector- and component-wise stochastic integrals with respect to a multidimensional semimartingale, we refer to Jacod (1979) and Protter (2003).

8.1.1 Standard Spot Market

We assume first that processes \(Z^1, \ldots, Z^k\) represent the spot prices of some traded assets. It is convenient to assume that \(Z^k\) (and thus also \(1/Z^k\)) follows a continuous, strictly positive semimartingale. We take \(Z^k\) as a benchmark security; in other words, we choose \(Z^k\) as the numeraire asset. Following the seminal paper of Harrison and Pliska (1981) (see also Harrison and Kreps (1979)), we say that an \(\mathbb{R}^k\)-valued predictable\(^1\) stochastic process \(\phi_t = (\phi^1_t, \ldots, \phi^k_t), t \in [0, T]\), is a self-financing (spot) trading strategy over time interval \([0, T]\) if the wealth process \(V(\phi)\), which equals

\[
V_t(\phi) \overset{\text{def}}{=} \phi_t \cdot Z_t = \sum_{i=1}^k \phi^i_t Z^i_t, \quad \forall \ t \in [0, T].
\]

satisfies \(V_t(\phi) = V_0(\phi) + G_t(\phi)\) for every \(t \leq T\), where \(G_t(\phi)\) stands for the gains process

\[
G_t(\phi) \overset{\text{def}}{=} \int_0^t \phi_u \cdot dZ_u, \quad \forall \ t \in [0, T].
\]

\(^1\) For the definition of a continuous-time predictable process, see Protter (2003) or Revuz and Yor (1999). Basically, predictability is a slight extension of the left-continuity of the sample paths of the process. In the case of the Itô integral with respect to continuous local martingales, it is actually enough to assume that the integrand is progressively measurable.