Polynomial C^2 Spline Surfaces Guided by Rational Multisided Patches

Kęstutis Karčiauskas[1] and Jörg Peters[2]

[1] Vilnius University, Vilnius 03225, Lithuania,
kestutis.karciauskas@maf.vu.lt,
WWW home page: http://www.mif.vu.lt/cs2/cagl/
[2] University of Florida, Gainesville, FL 32611, USA
jorg@cise.ufl.edu,
WWW home page: http://www.cise.ufl.edu/∼jorg/

Abstract. An algorithm is presented for approximating a rational multi-sided M-patch by a C^2 spline surface. The motivation is that the multi-sided patch can be assumed to have good shape but is in nonstandard representation or of too high a degree. The algorithm generates a finite approximation of the M-patch, by a sequence of patches of bidegree $(5, 5)$ capped off by patches of bidegree $(11, 11)$ surrounding the extraordinary point.
The philosophy of the approach is (i) that intricate reparametrizations are permissible if they improve the surface parametrization since they can be precomputed and thereby do not reduce the time efficiency at runtime; and (ii) that high patch degree is acceptable if the shape is controlled by a guiding patch.

1 Introduction

When constructing C^2 spline surfaces using a finite number of tensor-product Bézier patches such as [4, 11, 13, 7], the shape is often unsatisfactory near extraordinary points where more or less than four patches meet since the curvature is not evenly distributed or shows local extrema not implied by the surrounding data [10]. For subdivision schemes, the cause of shape artifacts has recently been analyzed and made explicit (see [9], [6]).

By contrast, rational multisided M-patches [5], joined smoothly to a surrounding B-spline complex, appear to consistently yield good shape. This paper does not verify the emperical observation of good shape but explores the technical challenge of how to transfer a good M-patch into a standard spline framework.

M-patches are rational and can therefore be represented as a collection of rational tensor-product Bézier subpatches. If the number of sides is 5 or 6, a variant of the M-patch can be represented as a collection of rational tensor-product Bézier patches of bidegree $(8, 8)$. But for a general m-sided M-patch, the bidegree is $(4(m-2), 4(m-2))$.

The idea pursued in this paper is to capture the shape of the M-patch with a C^2 approximation of moderate bidegree. We describe a finite approximation of the M-patch, by a sequence of patches of bidegree $(5, 5)$ but with patches adjacent to the extraordinary point of bidegree $(11, 11)$. A key point is the definition of reparametrizations that improve the surface parametrization when approximating the M-patch. These reparametrization, maps $\mathbb{R}^2 \to \mathbb{R}^2$, decompose the domain m-gon into C^2-connected

annuli or rings and a central piece. The surface approximation consists correspondingly of nested annuli and a final cap. Each annulus and the cap pick up second order Hermite data from the M-patch.

The bidegree of the central, extraordinary patches is still high, but all experiments with constructions of lower the bidegree have so far resulted in a considerably reduction of the surface quality. The paper focuses on the technical challenge of creating a C^2 surface that Hermite-interpolates a given M-patch.

One of the key ideas, construction of good reparametrizations and composition with a polynomial patch that determines the shape is a logical extension of similar ideas proposed in [11, 13, 7].

The paper is organized as follows.
Section 2 defines M-patches and the transition of the M-patch to an existing spline complex.
Section 3 defines the transitional reparametrizations and the patches of bidegree $(5, 5)$.
Section 4 describes the cap of patches of bidegree (11,11) that approximates the M-patch near the extraordinary point.

The exposition expects familiarity with standard representations of geometric design. The tensor-product Bernstein-Bézier and (uniform) B-spline representations (see [1, 3, 12]) have the form $\sum_{i=0}^{d_1}\sum_{j=0}^{d_2}\mathbf{c}_{ij}b_i^{d_1}(u)b_j^{d_2}(v)$, where b_i^d is the ith basis function. In the case of the Bernstein-Bézier form, $b_i^d(u) := \binom{d}{i}(1-u)^{d-i}u^i$. In particular, one can choose the domain as a unit square and then associate layers of coefficients $\mathbf{c}_{ij}$, $j \leq k$ with kth derivatives perpendicular to an edge of the square. In the following, we will often refer to *the three boundary layers* of a polynomial to mean the layers of coefficients $\mathbf{c}_{ij}$, $j \leq 3$ that determine position, first and second derivative across a boundary. Catmull-Clark subdivision [2] generalizes the refinement of bicubic uniform splines to control nets with nodes of arbtrary valence as illustrated in Fig. 1

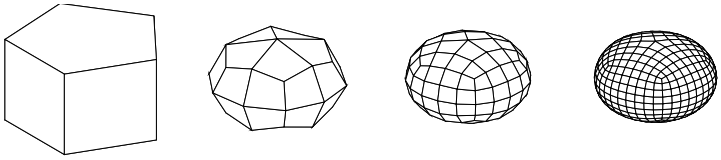

Fig. 1. Three steps of Catmull-Clark subdivision

2 M-patches and their tensor-border

In this section, we review the construction of rational multi-sided patches, called M-patches [5]. First, M-patches are defined. Then a control structure, called tensor-border, is defined that mimicks the behavior of tensor-product patches along the boundary. Such