Fast Closed Loop Control of the Navier-Stokes System

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**Summary.** We present a construction recipe for closed-loop feedback control of the time-dependent Navier-Stokes system. Its basic idea consists in approximately solving certain instantaneous optimization problems for the discrete-in-time dynamical system. Easy incorporation of control constraints is one key feature of the recipe. We state stabilizing properties of the controllers which we confirm by numerical tests.

1 Introduction

This research is devoted to the construction, numerical validation and stability analysis of nonlinear feedback control policies for the instationary Navier-Stokes system. The governing equations in the primitive setting are given by

\[
\begin{aligned}
y_t - \nu \Delta y + (y \cdot \nabla)y + \nabla p &= B u & \text{in } (0,T) \times \Omega, \\
\text{div } y &= 0 & \text{in } (0,T) \times \Omega, \\
y &= 0 & \text{on } (0,T) \times \partial \Omega, \\
y(0) &= \varphi & \text{in } \Omega.
\end{aligned}
\]

The control target is to match the given desired state \(z\) in the \(L^2\)-sense by adjusting the body force \(Bu\). In this context \(B\) denotes an abstract control extension operator and \(\Omega \subset \mathbb{R}^2\) denotes a bounded domain.

In this work we present recipes for the construction of nonlinear feedback control laws for the time-dependent Navier-Stokes system of the form

\[Bu = K(y)\]

and numerically illustrate their performance.

The construction principle works as follows. The uncontrolled Navier-Stokes system is discretized with respect to time. Then, at selected time instants an appropriate cost functional is approximately minimized with respect
to a stationary quasi-(Navier-)Stokes system, whose structure depends on the chosen time discretization method. The obtained control is used to steer the system to the next time instant, where the procedure is repeated. We note that this approach is related to model predictive control techniques, see [6].

**Main result**: Given a sufficiently smooth desired state \( z \), and a time discretization scheme for the Navier-Stokes system, the above described construction process can be regarded as time discretization of a closed loop feedback policy \( K \), i.e. with \( A \) denoting the Stokes operator and \( b(y) \) the nonlinearity of the Navier-Stokes equations we get the controlled system

\[
y_t + Ay + b(y) = K(y).
\]

Under certain assumptions on the initial states the controller \( K \) steers the Navier-Stokes system exponentially fast to \( z \). To be more precise, the solution of this system satisfies \( |y(t) - z(t)|_{H^1} \leq ce^{-\kappa t} \) with some positive constants \( c \) and \( \kappa \).

It turns out that instantaneous control [9, 11] is a special case of our approach. For applications of instantaneous control we refer to [1, 2, 3, 8, 10, 16, 17, 19], stability analysis of the method is presented in [9, 11]. Further contributions to closed loop control of the Navier-Stokes system can be found in [7], where linear force feedback control was applied to control the system. The analysis of special case of model predictive control of the Navier-Stokes equations can be found in [14, 15].

The paper is organized as follows. Section 2 contains the analytical preliminaries. In Section 3 we introduce the basic construction recipe which lead to certain discretized closed-loop control laws. These discrete laws are related to continuous closed-loop control laws, whose stability properties are also stated. Finally, in Section 5 numerical examples are presented, which confirm the theoretical results.

Throughout this work \( c \) and \( C \) denote global generic constants whose dependencies are mentioned when necessary.

## 2 Analytical preliminaries and time discretization

For given \( T > 0 \) let \( Q = (0, T) \times \Omega \), where \( \Omega \subset \mathbb{R}^2 \) is a bounded domain. We set \( V = \{ v \in H^1_0(\Omega)^2, \; \text{div} \; v = 0 \} \), \( H = \text{dom}_{L^2(\Omega)^2} \{ v \in C_0^\infty(\Omega)^2, \; \text{div} \; v = 0 \} \) and identify the Hilbert space \( H \) with its dual \( H' \). The dual space of \( V \) is defined to get a Gelfand-triple \( V \hookrightarrow H \hookrightarrow V' \). On \( H \) the common inner product is used, and \( V \) is endowed with the inner product

\[
(\varphi, \psi)_V = (\varphi', \psi')_H \quad \text{for} \; \varphi, \psi \in V.
\]

Moreover, with \( Z \) denoting a Hilbert space, \( L^p(Z) \) \((1 \leq p \leq \infty)\) denotes the space of measurable abstract functions \( \varphi : (0, T) \to Z \), which are \( p \)-integrable \((1 \leq p < \infty)\), or essentially bounded on \((0, T) \) \((p = \infty)\), respectively.