8 FFT-based Option Pricing

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8.1 Introduction

The Black-Scholes formula, one of the major breakthroughs of modern finance, allows for an easy and fast computation of option prices. But some of its assumptions, like constant volatility or log-normal distribution of asset prices, do not find justification in the markets. More complex models, which take into account the empirical facts, often lead to more computations and this time burden can become a severe problem when computation of many option prices is required, e.g. in calibration of the implied volatility surface. To overcome this problem Carr and Madan (1999) developed a fast method to compute option prices for a whole range of strikes. This method and its application are the theme of this chapter.

In Section 8.2, we briefly discuss the Merton, Heston, and Bates models concentrating on aspects relevant for the option pricing method. In the following section, we present the method of Carr and Madan which is based on the fast Fourier transform (FFT) and can be applied to a variety of models. We also consider briefly some further developments and give a short introduction to the FFT algorithm. In the last section, we apply the method to the three analyzed models, check the results by Monte Carlo simulations and comment on some numerical issues.

8.2 Modern Pricing Models

The geometric Brownian motion (GBM) is the building block of modern finance. In particular, in the Black-Scholes model the underlying stock price is
assumed to follow the GBM dynamics:

\[ dS_t = rS_t dt + \sigma S_t dW_t, \]  

(8.1)

which, applying Itô’s lemma, can be written as:

\[ S_t = S_0 \exp \left\{ \left( r - \frac{\sigma^2}{2} \right) t + \sigma W_t \right\}. \]  

(8.2)

The empirical facts, however, do not confirm model assumptions. Financial returns exhibit much fatter tails than the Black-Scholes model postulates, see Chapter 1. The common big returns that are larger than six-standard deviations should appear less than once in a million years if the Black-Scholes framework were accurate. Squared returns, as a measure of volatility, display positive autocorrelation over several days, which contradicts the constant volatility assumption. Non-constant volatility can be observed as well in the option markets where “smiles” and “skews” in implied volatility occur. These properties of financial time series lead to more refined models. We introduce three such models in the following paragraphs.

### 8.2.1 Merton Model

If an important piece of information about the company becomes public it may cause a sudden change in the company’s stock price. The information usually comes at a random time and the size of its impact on the stock price may be treated as a random variable. To cope with these observations Merton (1976) proposed a model that allows discontinuous trajectories of asset prices. The model extends (8.1) by adding jumps to the stock price dynamics:

\[ \frac{dS_t}{S_t} = r dt + \sigma dW_t + dZ_t, \]  

(8.3)

where \( Z_t \) is a compound Poisson process with a log-normal distribution of jump sizes. The jumps follow a (homogeneous) Poisson process \( N_t \) with intensity \( \lambda \) (see Chapter 14), which is independent of \( W_t \). The log-jump sizes \( Y_i \sim N(\mu, \delta^2) \) are i.i.d random variables with mean \( \mu \) and variance \( \delta^2 \), which are independent of both \( N_t \) and \( W_t \).