The Use of Energy Balance Relations for Validation of Gravity Field Models and Orbit Determination Results

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Abstract. There is a need for a proper validation procedure of gravity field solutions, especially for those high precise ones which are derived from the dedicated gravity field missions as CHAMP, GRACE and in future GOCE. In this paper the balance equations for the energy and for the energy exchange of a satellite orbiting around the Earth are proposed as analysis tool to validate gravity field solutions together with orbits derived with precise reduced dynamic or kinematic orbit determination procedures. The theoretical foundation of the analysis tool is presented as well as simulation results and applications to real orbits of a GPS satellite and of CHAMP based on various gravity field solutions. It is shown that the validation procedure can be used to detect deficiencies in the orbit modelling and in gravity field recovery results.

Keywords. Gravity field validation – Orbit determination validation – Energy balance relations – Jacobi-Integral

1 Introduction

The satellite missions CHAMP, GRACE and GOCE will provide unprecedented views of the Earth's gravity field and its changes with time. For many applications in the Earth sciences the new satellite solutions shall not be combined with available gravity field information, to provide consistent and unbiased satellite-only results of the gravity field. Therefore, a natural control of the recovery results by alternative data sets does not exist. Evidently, there is an urgent need for a proper validation tool for gravity field solutions of high precision. The validation and verification procedures based on comparisons with existing models in well-determined regions or based on orbit predictions may not fulfill the expectations for a rigorous validation: Available high-frequency solutions derived from satellite perturbations and altimetry and combined with terrestrial gravity field information are not consistent because of different datum references in space and time. Furthermore, the computation of gravity field solutions based on satellite techniques require the regularization of the large and ill-posed normal equations, which produces a bias in the results and can mislead the interpretation of the gravity field. On the other hand, comparisons on the basis of orbit computations by using a gravity field model are susceptible to numerical errors and the orbit residuals caused by incorrect force models may hide the real causes of deficiencies.

The energy exchange relations of the satellites' motion within the gravity field of the Earth seem to be useful for the validation and the consistency check of gravity field models and orbit determination results when applied in the sense of a forward computation process. One of the advantages is the more or less point wise application along the orbit, avoiding any error accumulation or instability effects. Besides the validation tasks the energy relations can represent the basis for a subsequent combination with alternative and additional gravity field information and may be used to calibrate measurement procedures.

The application of the energy integral for problems of satellite geodesy has been proposed since its very beginning (e.g., O'Keefe, 1959, Bjerhammar, 1967, Reigber, 1969). But the applications did not lead to convincing results because of the type of observations and the poor coverage of the satellite orbits with observations available at that time. The situation changed with the new type of homogeneous and dense data distributions as recently demonstrated e.g. by Jekeli (1999).

2 Gibbs function and Gibbs fundamental form

A physical system can be characterized by its Gibbs' function, a function which represents the total energy of the system (Falk and Ruppel, 1973). The Gibbs function is formulated in terms of "extensive" quantities $X_i$, $E(X_1,\ldots,X_n)$. In those cases where all quantities are well-known, the physical system is completely described, as well as the processes, in which the system can exchange dynamical quantities such as energy, linear momentum or angular momentum. The exchange of energy then occurs in various energy forms. Any energy form is defined by a pair of conjugate variables, the "intensive" and the "extensive" quantities. The exchange of energy is expressed by the change of the extensive variable. The product of an intensive quantity and the differential of an extensive variable results in the energy form. The change $dE$ of the total energy $E$ of the system is formulated by
Gibbs fundamental form:

\[ dE = \sum_{j=1}^{n} \xi_j dX_j . \]  

(1)

The intensive variables are derived as partial derivatives of Gibbs function with respect to the conjugate extensive variables

\[ \xi_j = \frac{\partial X_j}{\partial X} . \]  

(2)

The Gibbs function can be identified with the Hamiltonian function and the generalized impulses and coordinates with the extensive variables, while the time derivatives of the latter quantities correspond to the intensive variables. If we consider the simple case of a satellite, orbiting around a rigid Earth, and if we take into account only mechanical energy forms then the Gibbs function reads (Ilk, 2002, 2003),

\[ E = \frac{1}{2} p^2 + \frac{1}{2} L_E \cdot \omega_E + V(R, \Omega) = \text{const} , \]  

(3)

with the linear momentum \( p_x \) of the Earth, the angular momentum \( L_E \) of the Earth and \( T_E \) its inertia tensor. \( p \) is the linear momentum of the satellite referred to the geo center and \( M \) its mass. \( V(R, \Omega) \) is the potential energy of the satellite with its position \( R \) and the rotation vector \( \Omega \) of the Earth. Linear and angular momentum of the Earth can be considered as constants and included in the energy constant, so that it follows the relation,

\[ \hat{E} = \frac{1}{2} \frac{p^2}{M} + \hat{V}(R, \Omega) = \text{const} , \]  

(4)

which represents the well known energy balance for this simple case formulated with respect to an inertial system. In case of a rotating reference system an inertia field is involved. In case of a constant rotation expressed by the rotation vector \( \Omega \) the energy balance reads

\[ \hat{E} = \frac{1}{2} \frac{p^2}{M} + \hat{V}(R, \Omega) = \text{const} , \]  

(5)

where quantities referring to the rotating reference system are marked by an inverted comma. The potential energy \( V'(R') \) is related to the gravitational potential \( V''(R') \) of the Earth by the relation

\[ \hat{V}'(R') = -MV''(R') . \]  

(6)

The velocity of the satellite can be derived from the Gibbs function by differentiation with respect to the linear momentum \( (p' = P) \) due to (2), so that it holds:

\[ \hat{R}' = \frac{\partial \hat{E}}{\partial p'} = \frac{P'}{M} - \Omega \times R' = \hat{R} - \Omega \times R' , \]  

(7)

where \( \hat{R} \) is the velocity referred to the inertial system. Inserting this relation into formula (5) results in an energy balance relation, the so-called Jacobi integral,

\[ \hat{E}' = \frac{1}{2} \frac{R'^2}{M} - \frac{1}{2} (\Omega \times R')^2 - MV'(R') = \text{const} , \]  

(8)

and divided by the constant satellite mass \( M \),

\[ \hat{E}' = \frac{1}{2} \frac{R'^2}{2} - \frac{1}{2} (\Omega \times R')^2 - V'(R') = \text{const} . \]  

(9)

The rotational term represents the centrifugal potential and can be added to the gravitational potential to result in the gravity potential

\[ W'(R') = \frac{1}{2} (\Omega \times R')^2 + V'(R') , \]  

(10)

so that the (specific) energy balance can be written in the form

\[ \hat{E}' = \frac{1}{2} \frac{R'^2}{2} - W'(R') = \text{const} . \]  

(11)

### 3 Energy Integral for Satellite Motion

Unfortunately, the Gibbs function is not accessible in an easy way if the motion of an artificial satellite within the gravity field of the Earth has to be considered under real conditions. Because of the various rotational constituents of the Earth, such as daily rotation, polar motion and precession-nutation, the rotation vector \( \Omega \) is not constant. Furthermore, in the case of the satellite mission CHAMP but also GRACE, accelerations caused by surface forces such as air drag and solar pressure acting on the satellites are measured and it is not easy to formulate the corresponding functional constituents in the Gibbs function. If the surface accelerations are not observable in-situ then models for these kinds of forces can be used; again the corresponding energy functions are not available directly. Additional problems occur if the physical system has to include also Sun and Moon and the various time dependent and therefore non-conservative tidal constituents acting on the motion of the satellite. In this case and in case of additional non-conservative forces \( K' \) it is preferable to compute the energy contributions by the work \( \int A \) performed by the various forces along the satellite orbit within the time period \([t_0, t] \) (Löcher, unpublished manuscript):

\[ \hat{E}^{\int} = \int K' \cdot dR' = \int K' \cdot R' \cdot dt . \]  

(12)

Inserting the velocity according to equation (7) this formula reads

\[ \hat{E}^{\int} = \int K' \cdot dR' = \int K' \cdot R' \cdot \left( \frac{P'}{M} - \Omega \times R \right) \cdot dt \]  

(13)

The contribution of the centrifugal acceleration \( Z' \) to the energy balance is already included in the energy balance and corresponds to the rotational term in the Jacobi integral in form of formula (9). There is no contribution of the Coriolis acceleration \( C' \). In case of temporal variations of the rotation vector \( \Omega \) the contribution of the Eulerian acceleration,

\[ \hat{E}' = -M \hat{\Omega}' \times R' = -M \hat{\Omega} \times R , \]  

(14)

to the energy balance has to be taken into account as