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Introduction

In this book, we understand the solution of a decision problem as to choose "good" or "best" among a set of "alternatives," where we assume the existence of certain criteria, according to which the quality of the alternatives is measured. In this introductory chapter, we shall first give some examples and distinguish different types of decision problems. Informally, we shall understand optimization problems as mathematical models of decision problems. We introduce the concepts of decision (or variable) and criterion (or objective) space and mention different notions of optimality. Relations and cones are used to formally define optimization problems, and a classification scheme is introduced.

1.1 Optimization with Multiple Criteria

Let us consider the following three examples of decision problems.

Example 1.1. We want to buy a new car and have identified four models we like: a VW Golf, an Opel Astra, a Ford Focus and a Toyota Corolla. The decision will be made according to price, petrol consumption, and power. We prefer a cheap and powerful car with low petrol consumption. In this case, we face a decision problem with four alternatives and three criteria. The characteristics of the four cars are shown in Table 1.1 (data are invented).

How do we decide, which of the four cars is the "best" alternative, when the most powerful car is also the one with the highest petrol consumption, so that we cannot buy a car that is cheap as well as powerful and fuel efficient. However, we observe that with any one of the three criteria alone the choice is easy. □

Table 1.1. Criteria and alternatives in Example 1.1.

		Alternatives			
		VW	Opel	Ford	Toyota
Criteria	Price (1,000 Euros)	16.2	14.9	14.0	15.2
	Consumption $\left(\frac{l}{100km}\right)$	7.2	7.0	7.5	8.2
	Power (kW)	66.0	62.0	55.0	71.0

Example 1.2. For the construction of a water dam an electricity provider is interested in maximizing storage capacity while at the same time minimizing water loss due to evaporation and construction cost. A decision must be made on man months used for construction as well as mean radius of the lake, and also it must respect certain constraints such as minimal strength of the dam. Here, the set of alternatives (possible dam designs) allows infinitely many different choices. The criteria are functions of the decision variables to be maximized or minimized. The criteria are clearly in conflict: A dam with big storage capacity will certainly not involve small construction cost, for instance. □

Example 1.3. As a third example, we consider a mathematical problem with two criteria and one decision variable. The criteria or objective functions, which we want to minimize simultaneously over the nonnegative real line, are

$$f_1(x) = \sqrt{x+1} \quad \text{and} \quad f_2(x) = x^2 - 4x + 5 = (x-2)^2 + 1, \tag{1.1}$$

plotted in Figure 1.1. We want to solve the optimization problem

$$\text{“}\min_{x \geq 0}\text{”}(f_1(x), f_2(x)). \tag{1.2}$$

The question is, what are the “minima” and the “minimizers” in this problem? Note that again, for each function individually the corresponding optimization problem is easy: $x_1 = 0$ and $x_2 = 2$ are the (unique) minimizers of f_1 and f_2 on $x \in \mathbb{R} : x \geq 0$, respectively. □

The first two examples allow a first distinction of decision problems. Those decision problems with a countable number of alternatives are called *discrete*, others *continuous*. In this book, we will be concerned with both continuous and discrete problems.

Comparing Examples 1.1 and 1.3, another distinguishing feature of decision problems becomes apparent: In Example 1.1 the alternatives are explicitly