2

Efficiency and Nondominance

This chapter covers the fundamental concepts of efficiency and nondominance. We first present some fundamental properties of nondominated points and several existence results for nondominated points and efficient solutions in Section 2.1. Section 2.2 introduces ideal and nadir points as bounds on the set of nondominated solutions. Then we briefly review weakly and strictly efficient solutions in Section 2.3. The same section also includes a geometric characterization of the three optimality concepts, with some extensions for the case of weakly efficient solutions. Finally, in Section 2.4 we introduce several definitions of properly efficient solutions, important subsets of efficient solutions from a computational point of view and in applications, and investigate their relationships.

Most of the material in this chapter can be found in the two books Göpfert and Nehse (1990) and Sawaragi *et al.* (1985), where the results are presented in more generality. We will also refer to the original publications for the main results.

2.1 Efficient Solutions and Nondominated Points

In this chapter we consider multicriteria optimization problems of the class $\bullet/\mathrm{id}/(\mathbb{R}^p, \leq)$:

$$\begin{aligned} \min\ & (f_1(x), \dots, f_p(x)) \\ & \text{subject to } x \in \mathcal{X}. \end{aligned} \tag{2.1}$$

The image of the feasible set $\mathcal{X}$ under the objective function mapping f is denoted as $\mathcal{Y} := f(\mathcal{X})$. Let us formally repeat the definition of efficient solutions and nondominated points. Definition 2.1 also introduces the notion of dominance.

Definition 2.1. *A feasible solution $\hat{x} \in \mathcal{X}$ is called* efficient *or* Pareto optimal*, if there is no other $x \in \mathcal{X}$ such that $f(x) \leq f(\hat{x})$. If $\hat{x}$ is efficient, $f(\hat{x})$ is called* nondominated point*. If $x^1, x^2 \in \mathcal{X}$ and $f(x^1) \leq f(x^2)$ we say x^1* dominates *x^2 and $f(x^1)$* dominates *$f(x^2)$. The set of all efficient solutions $\hat{x} \in \mathcal{X}$ is denoted $\mathcal{X}_E$ and called the* efficient set*. The set of all nondominated points $\hat{y} = f(\hat{x}) \in \mathcal{Y}$, where $\hat{x} \in \mathcal{X}_E$, is denoted $\mathcal{Y}_N$ and called the* nondominated set*—.*

We have to remark that these notations are not unique in literature, unfortunately. Some authors use Pareto optimal for what we call efficient and efficient for what we call nondominated (e.g. this notation was used in the first edition of this book). The term noninferior solution has also been used. We will use the terms of Definition 2.1, but whenever consulting literature, the reader should check the definitions the respective author adopts.

Several other, equivalent, definitions of efficiency are frequently used, and we shall often refer to the one which is best suited in a given context. In particular, $\hat{x}$ is efficient if

1. there is no $x \in \mathcal{X}$ such that $f_k(x) \leq f_k(\hat{x})$ for $k = 1, \ldots, p$ and $f_i(x) < f_i(\hat{x})$ for some $i \in \{1, \ldots, k\}$;
2. there is no $x \in X$ such that $f(x) - f(\hat{x}) \in -\mathbb{R}^p_{\geqq} \setminus \{0\}$;
3. $f(x) - f(\hat{x}) \in \mathbb{R}^p \setminus \left\{-\mathbb{R}^p_{\geqq} \setminus \{0\}\right\}$ for all $x \in \mathcal{X}$;
4. $f(\mathcal{X}) \cap \left(f(\hat{x}) - \mathbb{R}^p_{\geqq}\right) = \{f(\hat{x})\}$;
5. there is no $f(x) \in f(\mathcal{X}) \setminus \{f(\hat{x})\}$ with $f(x) \in f(\hat{x}) - \mathbb{R}^p_{\geqq}$;
6. $f(x) \leqq f(\hat{x})$ for some $x \in \mathcal{X}$ implies $f(x) = f(\hat{x})$.

With the exception of the last, these definitions can be illustrated graphically. Definition 2.1 and equivalent definitions 1., 4., and 5. consider $f(\hat{x})$ and check for images of feasible solutions to the left and below (in direction of $-\mathbb{R}^p_{\geqq}$) of that point. See the left part of Figure 2.1. In equivalent definitions 2. and 3., through $f(x) - f(\hat{x})$, the set $\mathcal{Y} = f(\mathcal{X})$ is translated so that the origin coincides with $f(\hat{x})$, and the intersection of the translated set $\mathcal{Y}$ with the negative orthant is checked. This intersection contains only $f(\hat{x})$ if $\hat{x}$ is efficient. See the right part of Figure 2.1.

The first questions we discuss are the existence and the properties of the efficient set $\mathcal{X}_E$ and the nondominated set $\mathcal{Y}_N$. It is convenient to consider $\mathcal{Y}_N$ first, and then use properties of f to derive results on $\mathcal{X}_E$. So let $\mathcal{Y} \subset \mathbb{R}^p$ be a set. According to our definitions, $\hat{y} \in \mathcal{Y}$ is nondominated, if there is no $y \in \mathcal{Y}$ such that $y \leq \hat{y}$.

First we show by means of an example that even for convex sets $\mathcal{X}$ and $\mathcal{Y}$ the efficient set $\mathcal{X}_E$ and the nondominated set set $\mathcal{Y}_N$ might be empty or