# 3

# The Weighted Sum Method and Related Topics

In this chapter we will investigate to what extent an MOP

$$\min_{x \in \mathcal{X}} (f_1(x), \dots, f_p(x)) \tag{3.1}$$

of the Pareto class

$$(\mathcal{X}, f, \mathbb{R}^p)/\mathrm{id}/(\mathbb{R}^p, \leq)$$

can be solved (i.e. its efficient solutions be found) by solving single objective problem problems of the type

$$\min_{x \in \mathcal{X}} \sum_{k=1}^{p} \lambda_k f_k(x), \tag{3.2}$$

which in terms of the classification of Section 1.5 is written as

$$(\mathcal{X}, f, \mathbb{R}^p)/\langle \lambda, \cdot \rangle/(\mathbb{R}, \leq), \tag{3.3}$$

where $\langle \lambda, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^p$. We call the single objective (or scalar) optimization problem (3.2) a *weighted sum scalarization* of the MOP (3.1).

As in the previous chapter, we will usually look at the objective space $\mathcal{Y}$ first and prove results on the relationships between (weakly, properly) nondominated points and values $\sum_{k=1}^{p} \lambda_k y_k$. From those, we can derive results on the relationships between $\mathcal{X}_{(w,p)E}$ and optimal solutions of (3.2).

We use these results to prove Fritz-John and Kuhn-Tucker type optimality conditions for (weakly, properly) efficient solutions (Section 3.3). Finally, we investigate conditions that guarantee that nondominated and efficient sets are connected (Section 3.4).

Let $\mathcal{Y} \subset \mathbb{R}^p$. For a fixed $\lambda \in \mathbb{R}^p_{\geqq}$ we denote by

$$\mathcal{S}(\lambda, \mathcal{Y}) := \left\{ \hat{y} \in \mathcal{Y} : \ \langle \lambda, \hat{y} \rangle = \min_{y \in \mathcal{Y}} \langle \lambda, y \rangle \right\} \tag{3.4}$$

the set of optimal points of $\mathcal{Y}$ with respect to $\lambda$.

Figure 3.1 gives an example of a set $\mathcal{S}(\lambda, \mathcal{Y})$ consisting of two points $y^1$ and $y^2$. These points are the intersection points of a line $\{y \in \mathbb{R}^p : \langle \lambda, y \rangle = \hat{c}$. Obviously, $y^1$ and $y^2$ are nondominated. Considering $c$ as a parameter, and the family of lines $\{y \in \mathbb{R}^p : \langle \lambda, y \rangle = c\}$, we see that in Figure 3.1 $\hat{c}$ is chosen as the smallest value of $c$ such that the intersection of the line with $\mathcal{Y}$ is nonempty.

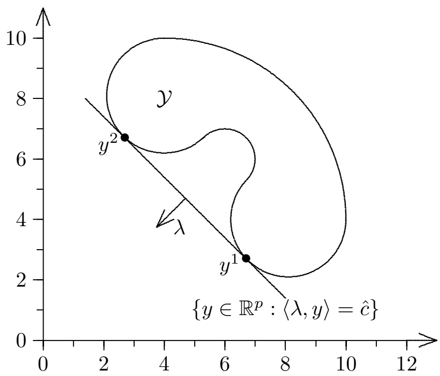


**Fig. 3.1.** A set $\mathcal{S}(\lambda, \mathcal{Y})$.

Graphically, to find $\hat{c}$ we can start with a large value of the parameter $c$ and translate the line in parallel towards the origin as much as possible while keeping a nonempty intersection with $\mathcal{Y}$. Analytically, this means finding elements of $\mathcal{S}(\lambda, \mathcal{Y})$. The obvious questions are:

1. Does this process always yield nondominated points? (Is $\mathcal{S}(\lambda, \mathcal{Y}) \subset \mathcal{Y}_N$?) and
2. if so, can all nondominated points be detected this way? (Is $\mathcal{Y}_N \subset \cup_{\lambda \in \mathbb{R}^p_{\geq}} \mathcal{S}(\lambda, \mathcal{Y})$?)

Note that due to the definition of nondominated points, we have to consider nonnegative weighting vectors $\lambda \in \mathbb{R}^p_{\geq}$ only. However, the distinction between nonnegative and positive weights turns out to be essential. Therefore we distinguish optimal points of $\mathcal{Y}$ with respect to nonnegative and strictly positive weights, and define