4

Scalarization Techniques

The traditional approach to solving multicriteria optimization problems of the Pareto class is by scalarization, which involves formulating a single objective optimization problem that is related to the MOP

$$\min_{x \in \mathcal{X}} (f_1(x), \ldots, f_p(x)) \tag{4.1}$$

by means of a real-valued scalarizing function typically being a function of the objective functions of the MOP (4.1), auxiliary scalar or vector variables, and/or scalar or vector parameters. Sometimes the feasible set of the MOP is additionally restricted by new constraint functions related to the objective functions of the MOP and/or the new variables introduced.

In Chapter 3 we introduced the "simplest" method to solve multicriteria problems, the weighted sum method, where we solve

$$\min_{x \in \mathcal{X}} \sum_{k=1}^{p} \lambda_k f_k(x). \tag{4.2}$$

The weighted sum problem (4.2) uses the vector of weights $\lambda \in \mathbb{R}^p_{\geq}$ as a parameter. We have seen that the method enables computation of the properly efficient and weakly efficient solutions for convex problems by varying λ. The following Theorem summarizes the results.

Theorem 4.1. *1. Let $\hat{x} \in \mathcal{X}$ be an optimal solution of* (4.2). *The following statements hold.*

- *If $\lambda > 0$ then $\hat{x} \in \mathcal{X}_{pE}$.*
- *If $\lambda \geq 0$ then $\hat{x} \in \mathcal{X}_{wE}$.*
- *If $\lambda \geq 0$ and $\hat{x}$ is a unique optimal solution of* (4.2) *then $\hat{x} \in \mathcal{X}_{sE}$.*

2. Let $\mathcal{X}$ be a convex set and $f_k, k = 1, \ldots, p$ be convex functions. Then the following statements hold.

- *If* $\hat{x} \in \mathcal{X}_{pE}$ *then there is some* $\lambda > 0$ *such that* $\hat{x}$ *is an optimal solution of* (4.2).
- *If* $\hat{x} \in \mathcal{X}_{wE}$ *then there is some* $\lambda \geq 0$ *such that* $\hat{x}$ *is an optimal solution of* (4.2).

For nonconvex problems, however, it may work poorly. Consider the following example.

Example 4.2. Let $\mathcal{X} = \{x \in \mathbb{R}^2_{\geqq} : x_1^2 + x_2^2 \geq 1\}$ and $f(x) = x$. In this case $\mathcal{X}_E = \{x \in \mathcal{X} : x_1^2 + x_2^2 = 1\}$, yet $\hat{x}^1 = (1,0)$ and $\hat{x}^2 = (1,0)$ are the only feasible solutions that are optimal solutions of (4.2) for any $\lambda \geq 0$.

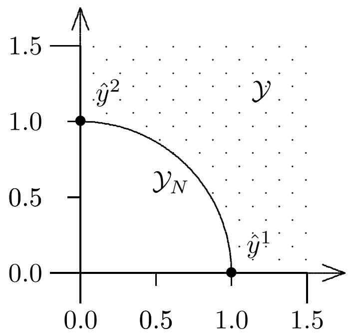

Fig. 4.1. The weighted sum method fails for nonconvex problems.

□

In this chapter we introduce some other scalarization methods, which are also applicable when $\mathcal{Y}$ is not $\mathbb{R}^p_{\geqq}$-convex.

4.1 The ε-Constraint Method

Besides the weighted sum approach, the ε-constraint method is probably the best known technique to solve multicriteria optimization problems. There is no aggregation of criteria, instead only one of the original objectives is minimized, while the others are transformed to constraints. It was introduced byHaimes *et al.* (1971), and an extensive discussion can be found in Chankong and Haimes (1983).

We substitute the multicriteria optimization problem (4.1) by the ε-constraint problem