5

Other Definitions of Optimality – Nonscalarizing Methods

The concept of efficiency and its variants are by far the most important definitions of optimality in multicriteria optimization. Their extensive coverage in Chapters 2 to 4 reflects this fact. But as we have seen in Chapter 1 with the discussion of orders and the classification of multicriteria problems this is not the end of the story. Other choices of orders and model maps give rise to different classes of multicriteria optimization problems. In this chapter we shall discuss some of these. Specifically we address lexicographic optimality, max-ordering optimality, and finally a combination of the two, lexicographic max-ordering optimality. Lexicographic max-ordering defines a class of problems with many interesting features. Of particular interest will be the relationships between optimal solutions of these problems and efficient solutions. In this way they can be seen as nonscalarizing methods for finding efficient solutions. We do not study these problems out of curiosity about their theory, however.

Lexicographic optimization problems arise naturally when conflicting objectives exist in a decision problem but for reasons outside the control of the decision maker the objectives have to be considered in a hierarchical manner. Weber *et al.* (2002) describe the optimization of water resources planning for Lake Verbano (Lago Maggiore) in northern Italy. The goal is to determine an optimal policy for the management of the water supply over some planning horizon. The objectives are to maximize flood protection, minimize supply shortage for irrigation, and maximization of electricity generation. This order of objectives is prescribed by law, so that the problem indeed has a lexicographic nature. The actual formulation of the problem is via stochastic dynamic programming, which is beyond the scope of this book, and we omit it.

A common application of max-ordering problems is location planning. Ehrgott (2002) describes the problem of locating rescue helicopters in South Tyrol, Italy. The objective in this problem is to minimize the distance be-

tween potential accident sites and the closest helicopter location. In order to minimize worst case response times in an emergency, the problem can be formulated as follows. Let $x^h = (x_1^h, x_2^h), h \in \mathcal{H}$ denote variables that define helicopter locations and $(a_1^k, a_2^k), k \in 1, \ldots, p$ the potential emergency sites. Optimal helicopter locations are found by solving

$$\min_{x \in \mathbb{R}^{2|\mathcal{H}|}} \max_{k \in 1,\ldots,p} f_k(x)$$

where $f_k(x)$ is defined as

$$f_k(x) = \min_{h \in \mathcal{H}} w_k ||x^h - a^k||_2.$$

Georgiadis *et al.* (2002) describe the problem of picking routes and associated route bandwidth in a computer network so that bandwidth request is satisfied and the network is in a balanced state, i.e. the bandwidth allocation results in an even spreading of the load to various links of the network. They formulate this problem as a lexicographic max-ordering network flow problem. Variables x_{ij} denote the load on links ij. Let $C_{ij}(x_{ij})$be a function that describes the link cost and $b(i)$ be the bandwidth demand at a node of the network. Then the balanced bandwidth allocation problem is

$$\begin{aligned} &\min \operatorname{sort}(C_{ij}(x_{ij})) \\ \text{subject to } &\sum_j x_{ij} - \sum_j x_{ji} = b(i), \quad i \in N \\ &x_{ij} \geq 0. \end{aligned}$$

These examples should give an indication that the lexicographic, max-ordering, and lexicographic max-ordering classes are very relevant for practical applications.

Before we start our investigations, we state one general assumption. Throughout this chapter we shall assume that the single objective optimization problems $\min_{x \in \mathcal{X}} f_k(x)$ have optimal solutions for $k = 1, \ldots, p$ and that $\mathcal{X}_E \neq \emptyset$, unless stated otherwise.

5.1 Lexicographic Optimality

In lexicographic optimization we consider the lexicographic order when comparing objective vectors in criterion space. As for efficiency, the model map is the identity map, so in terms of classification we deal with problems of the class $(\bullet/\text{id}/(\mathbb{R}^p, <_{\text{lex}}))$. An optimal solution $\hat{x}$ of such a problem is called lexicographically optimal and $f(\hat{x})$ is a lexicographically minimal vector in $\mathcal{Y} = f(\mathcal{X})$.