6

Introdcution to Multicriteria Linear Programming

This chapter commences the second part of this book, in which we focus on multicriteria problems with linear and combinatorial structures, i.e. multiobjective linear programming and multiobjective combinatorial optimization.

We give an example from the design of radiotherapy treatment plans to show that multiobjective linear programming has important applications. We repeat the main definitions of multicriteria optimization and summarize the main results from linear programming to make this part of the book self-contained. We apply some of the general results proved in Chapters 2 and 3 and show how to use parametric linear programming to solve linear programs with two objectives. We also prove the main theorem of linear programming, which states that all efficient solutions are properly efficient. Adding the convexity of linear problems this means that all efficient solutions can be characterized by weighted sum scalarization.

Example 6.1. The goal of radiation therapy in the treatment of cancer is to destroy a tumour by damaging the DNA of cancerous cells, thereby rendering them incapable of reproduction. This is done by focusing intensity modulated beams on the patient from a number of beam directions. Intensity modulation is achieved by a mechanical device called multileaf collimator. It essentially allows subdividing beams into sub-beams in a rectangular grid pattern so that intensity of each individual sub-beam can be decided separately. Given the beam directions, an intensity map defines the intensity of radiation of each sub-beam of all beam directions. The intensity map has to be determined according to a treatment prescription, which can take the form of lower and upper bounds on the radiation dose delivered to the tumour as well as upper bounds on the radiation dose delivered to critical structures (such as healthy organs) and normal tissue.

Radiation dose distribution in the body depends on intensity of radiation beams in a linear fashion. Let $x \in \mathbb{R}^n$ be a vector describing an intensity map, where n is the total number of sub-beams. The patient body is discretized into m dose points according to magnetic resonance imaging (MRI) or computed tomography (CT) scans. The dose delivered to the dose points is then Ax, where A is a $m \times n$ matrix. Assuming that we have l critical structures, we can partition the rows of A according to the set of dose points in the tumour $\mathcal{T}$, in a critical structure $\mathcal{S}_i, i = 1, \ldots, l$, or in normal tissue $\mathcal{N}$ and form submatrices $A_\mathcal{T}$, $A_{\mathcal{S}_i}$, $A_\mathcal{N}$ accordingly. Let $l_\mathcal{T}$ denote the prescribed tumouricidal dose, $u_\mathcal{T}$ be an upper bound on the dose in the tumour, $u_{\mathcal{S}_i}$ be upper bounds on the dose in critical structure i, and $u_\mathcal{N}$ be an upper bound on the dose in normal tissue. We assume that these bounds apply to every dose point in the tumour, critical structure, and normal tissue, respectively.

Ideally, we would like to design a treatment that delivers a uniform dose of $l_\mathcal{T}$ to the tumour and no dose at all to critical structures and normal tissue. Since this is usually physically impossible we have to accept some underdosing $z_\mathcal{T}$ in the tumour or overdosing $z_{\mathcal{C}_i}, i = 1, \ldots, l$ and $z_\mathcal{N}$ in critical structures and normal tissue. Naturally, the values of $z_\mathcal{T}, z_{\mathcal{S}_1}, \ldots, z_{\mathcal{C}_l}$ should be kept as small as possible.

We can therefore describe the problem via the following multiobjective optimization problem Holder (2004), where e is a vector of ones of appropriate dimension.

$$
\begin{array}{lrl}
\min & (z_\mathcal{T}, z_{\mathcal{S}_1}, \ldots, z_{\mathcal{S}_l}, z_\mathcal{N}) & \\
\text{subject to} & A_\mathcal{T} x + z_\mathcal{T} e \geq l_\mathcal{T} & \\
& A_\mathcal{T} x \leq u_\mathcal{T} & \\
& A_{\mathcal{S}_i} x - z_{\mathcal{S}_i} e \leq u_{\mathcal{S}_i} & i = 1, \ldots, l \\
& A_\mathcal{N} x - z_\mathcal{N} e \leq u_\mathcal{N} & \\
& z_{\mathcal{S}_i} \geq -u_{\mathcal{S}_i} & i = 1, \ldots, l \\
& z_\mathcal{N} \geq 0 & \\
& x \geqq 0. &
\end{array}
$$

In this model, the goal is to find efficient solutions $(x, z) \in \mathbb{R}^{n+l+2}$ such that the maximal underdosing of any tumour dose point and the maximal overdosing of any critical structure and any normal tissue dose point is simultaneously minimized. □

6.1 Notation and Definitions

A multiobjective linear program (MOLP) is a special case of the multiobjective program