7

A Multiobjective Simplex Method

An MOLP with two objectives can be conveniently solved using the parametric Simplex method presented in Algorithm 6.2. With three or more objectives, however, this is no longer possible because we deal with at least two parameters in the objective function $c(\lambda)$.

7.1 Algebra of Multiobjective Linear Programming

In this section we consider the general MOLP

$$\begin{aligned} \min \quad & Cx \\ \text{subject to } & Ax = b \\ & x \geqq 0. \end{aligned} \tag{7.1}$$

For $\lambda \in \mathbb{R}^p_>$ we denote by LP(λ) the weighted sum linear program

$$\min \left\{ \lambda^T Cx : Ax = b, x \geqq 0 \right\}. \tag{7.2}$$

We use the notation $\bar{C} = C - C_{\mathcal{B}} A_{\mathcal{B}}^{-1} A$ for the reduced cost matrix with respect to basis $\mathcal{B}$ and $R := \bar{C}_{\mathcal{N}}$ for the nonbasic part of the reduced cost matrix. Note that $\bar{C}_{\mathcal{B}} = 0$ according to (6.15) and is therefore uninteresting. Proofs in this section will make use of Theorem 6.11. These results are multicriteria analogies of well known linear programming results, or necessary extensions to cope with the increased complexity of multiobjective compared to single objective linear programming.

Lemma 7.1. *If $\mathcal{X}_E \neq \emptyset$ then $\mathcal{X}$ has an efficient basic feasible solution.*

Proof. By Theorem 6.11 there is some $\lambda \in \mathbb{R}^p_>$ such that $\min_{x \in \mathcal{X}} \lambda^T Cx$ has an optimal solution. But by Theorem 6.13 the LP(λ) $\min_{x \in \mathcal{X}} \lambda^T Cx$ has a optimal basic feasible solution solution, which is an efficient solution of the MOLP by Theorem 6.6. □

Lemma 7.1 justifies the definition of an efficient basis.

Definition 7.2. *A feasible basis $\mathcal{B}$ is called* efficient basis *if $\mathcal{B}$ is an optimal basis of LP(λ) for some $\lambda \in \mathbb{R}^p_>$.*

We now look at pivoting among efficient bases. We say that a pivot is a *feasible pivot* if the solution obtained after the pivot step is feasible, even if the pivot element $\tilde{A}_{rs} < 0$.

Definition 7.3. *Two bases $\mathcal{B}$ and $\hat{\mathcal{B}}$ are called* adjacent *if one can be obtained from the other by a single pivot step.*

Definition 7.4. *1. Let $\mathcal{B}$ be an efficient basis. Variable $x_j, j \in \mathcal{N}$ is called* efficient nonbasic variable *at $\mathcal{B}$ if there exists a $\lambda \in \mathbb{R}^p_>$ such that $\lambda^T R \geqq 0$ and $\lambda^T r^j = 0$, where r^j is the column of R corresponding to variable x_j.*
2. Let $\mathcal{B}$ be an efficient basis and let x_j be an efficient nonbasic variable. Then a feasible pivot from $\mathcal{B}$ with x_j entering the basis is called an efficient pivot *with respect to $\mathcal{B}$ and x_j.*

The system $\lambda^T R \geqq 0, \lambda^T r^j = 0$ is the general form of the equations we used to compute the critical λ values in parametric linear programming that were used to derive (6.23): We chose s such that $\bar{c}(\lambda) \geq 0, \bar{c}(\lambda)_s = 0$.

Proposition 7.5. *Let $\mathcal{B}$ be an efficient basis. There exists an efficient nonbasic variable at $\mathcal{B}$.*

Proof. Because $\mathcal{B}$ is an efficient basis there exists $\lambda > 0$ such that $\lambda^T R \geqq 0$. Thus the set $\mathcal{L} := \{\lambda > 0 : \lambda^T R \geqq 0\}$ is not empty. We have to show that there is $\lambda \in \mathcal{L}$ and $j \in \mathcal{N}$ such that $\lambda^T r^j = 0$.

First we observe that there is no column r of R such that $r \leq 0$. There also must be at least one column with positive and negative elements, because of the general assumption (6.2). Now let $\lambda^* \in \mathcal{L}$. In particular ${\lambda^*}^T \geqq 0$. Let $\lambda' \in \mathbb{R}^p_>$ be such that $\mathcal{I} := \{i \in \mathcal{N} : \lambda'^T r^j < 0\} \neq \emptyset$. Such a λ must exist, because R contains at least one negative entry.

We define $\phi : \mathbb{R} \to \mathbb{R}^{|\mathcal{N}|}$ by

$$\phi_i(t) := (t{\lambda^*}^T + (1-t)\lambda'^T)r^i, i \in \mathcal{N}.$$

Thus, $\phi(0) = \lambda'^T R$ and $\phi(1) = {\lambda^*}^T R \geqq 0$. For each $i \in \mathcal{N} \setminus \mathcal{I}$ we have that $\phi_i(t) \geq 0$ for all $t \in [0,1]$. For all $i \in \mathcal{I}$ there exists some $t_i \in [0,1]$ such that

$$\phi_i(t) \begin{cases} < 0, t \in [0, t_i) \\ = 0, t = t_i \\ \geq 0, t \in [t_i, 1]. \end{cases}$$

With $t^* := \max\{t_i : i \in \mathcal{I}\}$ we have that $\phi_i(t^*) \geqq 0$ and $\phi_i(t^*) = 0$ for some $i \in \mathcal{I}$. Thus $\hat{\lambda} := t\lambda^* + (1-t)\lambda' \in \mathcal{L}$ and the proof is complete. □