9

Multiobjective Versions of Polynomially Solvable Problems

9.1 Algorithms for the Shortest Path Problem

This section is about the shortest path problem with multiple objectives. Let $\mathcal{G} = (\mathcal{V}, \mathcal{A})$ be a directed graph (or digraph) with $|\mathcal{V}| = m$ nodes (or vertices) and $|E| = n$ arcs. Let $c : \mathcal{A} \to \mathbb{Z}^p$ be a cost function on the arcs. In this section we consider the problems of finding efficient paths from a specified node s to another specified node t, or from node s to all other nodes of $\mathcal{G}$ with sum objectives. We will not discuss the problem of finding efficient paths between all pairs of nodes. This problem can always be solved by solving m shortest path problems with fixed starting node s and has not been addressed in the multicriteria literature as a distinct problem.

We show that the problem with fixed s and t is difficult in terms of finding and counting efficient solutions and that it is intractable. The algorithms we present are generalizations of the well known label-setting and label-correcting algorithms for single objective shortest path problems. At the end of this section we present a ranking algorithm for the biobjective shortest path problem. We present a generalization of this algorithm that can be used as a prototype to solve any MOCO problem.

Definition 9.1. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{A})$ be a directed graph. Let $\mathcal{V} = \{v_1, \ldots, v_m\}$ and $\mathcal{A} = \{a_1, \ldots, a_n\}$ where $a_i = (v_{j_1}, v_{j_2})$ is an ordered pair of vertices.*

1. *A* path *P is a sequence of nodes and arcs $(v_{i_1}, a_{i_1}, v_{i_2}, \ldots, v_{i_{r-1}}, a_{i_{r-1}}, v_{i_r})$ such that either $a_{i_l} = (v_{i_l}, v_{i_{l+1}})$ or $a_{i_l} = (v_{i_{l+1}}, v_{i_l})$. In a* directed path *only the former is possible.*
2. *A* simple path *is a path without repetition of vertices. A* simple directed path *is a directed path without repetition of vertices.*
3. *A* cycle *C is a simple path together with the arc (v_{i_r}, v_{i_1}) or (v_{i_1}, v_{i_r}). A* directed cycle *is a directed simple path together with the arc (v_{i_r}, v_{i_1})*

We will often identify a (directed) path by its sequence of vertices $(v_{i_1}, \ldots, v_{i_r})$. Let $s, t \in \mathcal{V}$ be two vertices and let $\mathcal{P}$ denote the set of all directed paths with $v_1 = s$ and $v_r = t$. Let $c : \mathcal{A} \to \mathbb{Z}^p$ be a cost function. The multiobjective shortest path problem is to find all efficient directed paths from s to t, i.e.

$$\min_{P \in \mathcal{P}} \left\{ \sum_{a \in P} c(a) \right\}. \tag{9.1}$$

Theorem 9.2 (Serafini (1986)). *The bicriterion shortest path problem* (9.1) *is $\mathcal{NP}$-complete and $\#\mathcal{P}$-complete in acyclic digraphs.*

Proof. The decision version of (9.1) is: Given $b \in \mathbb{Z}^2$, $\mathcal{G} = (\mathcal{V}, \mathcal{A})$, and $s, t \in \mathcal{V}$ does there exist a path P from s to t in $\mathcal{G}$ such that $\sum_{a \in P} c(a) \leqq k$. This problem is clearly in $\mathcal{NP}$. We give a parsimonious transformation 0-1 KNAPSACK $\propto_p (\mathcal{P}, 2\text{-}\sum, \mathbb{Z}^2)/\operatorname{id}/(\mathbb{Z}^2, \leq)$. Given an instance a^1, a^2, b_1, b_2 of 0-1 KNAPSACK we construct an instance of the shortest path problem as follows. Let

$$\begin{aligned}
\mathcal{V} &:= \{v_0, \ldots, v_n\}, \\
s &:= v_0, \\
t &:= v_n, \\
\mathcal{A} &:= \{(v_{i-1}, v_i) : i = 1, \ldots, n\} \cup \{(v_{i-1}, v_i)' : i = 1, \ldots, n\}, \\
c^1(a) &:= \begin{cases} a_i^1 & \text{if } a = (v_{i-1}, v_i) \\ 0 & \text{if } a = (v_{i-1}, v_i)', \end{cases} \\
c^2(a) &:= \begin{cases} 0 & \text{if } a = (v_{i-1}, v_i) \\ a_i^2 & \text{if } a = (v_{i-1}, v_i)'. \end{cases}
\end{aligned}$$

The graph defined here is shown in Figure 9.1.

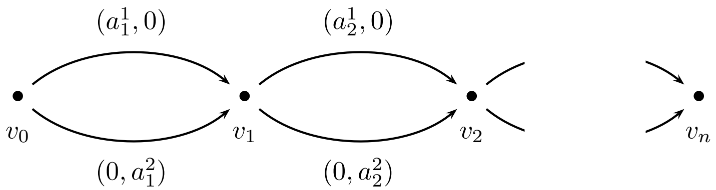

Fig. 9.1. Graph in the proof of Theorem 9.2.

Let $P \in \mathcal{P}$ be a path from s to t. Then