Numerical Transform Inversion for Autocorrelations of Waiting Times

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Abstract. The generating function of the autocorrelations of successive waiting times in a stationary M/G/1 or in a stationary GI/M/1 system can be expressed in terms of the probability generating function of the number of customers served in a busy period. The latter function is only implicitly determined as a solution to a functional equation. More explicit expressions have been obtained with the aid of Lagrange's theorem on the reversion of power series, but they involve increasingly higher order derivatives of a function which comprises several Laplace-Stieltjes transforms. A recently discovered substitution method for contour integrals allows the numerical inversion of an implicitly determined generating function without the numerical solution of the functional equation for many complex values.

1 Introduction

Autocorrelations of waiting times in queueing systems are useful in determining the variance of the mean of a sample of successive waiting times and provides an indication of how long a simulation should be run. They also give an indication how long it takes for a given backlog of work to fade away. The generating functions of the autocorrelations of the waiting times in stationary M/G/1 and GI/M/1 systems as determined by Blomqvist [5] and Pakes [8], respectively, involve the probability generating function (PGF) of the distribution of the number of customers served in a busy period. The latter functions are only implicitly determined as solutions to functional equations. Standard methods for the numerical inversion of generating functions require the values of these functions at many complex arguments, cf. Abate & Whitt [1,2]. A recent substitution method for contour integrals described in Blanc [3,4] allows the numerical inversion of implicitly determined generating functions without the need for numerical solution of the functional equations. This provides an efficient way to compute the autocorrelations of the waiting times for these systems.

Section 2 provides a short summary of a standard method for the numerical inversion of generating functions. Section 3 contains some general properties of waiting times in stationary GI/G/1 systems and introduces some notations. Section 4 deals with the derivation of an alternative contour integral for the numerical inversion of the GF of the series of autocorrelations of the waiting times in stationary M/G/1 systems. Section 5 is devoted to

a similar substitution, but for stationary GI/M/1 systems. The latter two sections contain several examples. Some final remarks on the accuracy of the method can be found in Sect. 6.

2 Numerical inversion of generating functions

The terms of a sequence of real numbers $\{g_k;\ k = 0, 1, 2, \ldots\}$ with $|g_k| \leq 1$ for all k can be recovered from its generating function (GF) by means of a contour integral in the complex plane over a circle around the origin with radius r, $0 < r < 1$:

$$G(z) \doteq \sum_{k=0}^{\infty} g_k z^k, \quad |z| < 1, \qquad g_k = \frac{1}{2\pi i} \oint_{|z|=r} G(z) \, \frac{dz}{z^{k+1}}, \quad k = 0, 1, \ldots . \tag{1}$$

The contour integral can be converted into an integral over a real interval by means of the substitution $z = re^{iu}$ and by some symmetry properties of the GF: for $0 < r < 1,\ k = 0, 1, \ldots,$

$$g_k = \frac{1}{\pi r^k} \int_0^{\pi} [\cos(ku)\ \Re G(re^{iu}) + \sin(ku)\ \Im G(re^{iu})]\, du\ ; \tag{2}$$

here, $i = \sqrt{-1}$ and $\Re z$ ($\Im z$) denotes the real (imaginary) part of a complex number z. The case $k = 0$ is simple: $g_0 = G(0)$. For $k > 0$, Abate & Whitt [1] describe the following method for evaluating the above type of integrals with a prescribed accuracy of, say, ε. Application of the trapezoidal rule with a step size of π/k to (2) yields, for $k = 1, 2, \ldots,$

$$g_k \approx \frac{1}{kr^k} \left[\frac{1}{2}\{G(r) + (-1)^k G(-r)\} + \sum_{j=1}^{k-1} (-1)^j\, \Re G(re^{ij\pi/k}) \right], \tag{3}$$

while the prescribed accuracy and an upper bound on the discretization error lead to the choice of $r = \sqrt[2k]{\varepsilon},\ k = 1, 2, \ldots$; to avoid roundoff problems, approximately $\frac{3}{2}\gamma$-digit precision is required to obtain $\varepsilon = 10^{-\gamma}$ accuracy. The numerical experiments for this paper have been performed with $\gamma = 10$ and about 16-digit precision.

3 Successive waiting times

The starting point for the study of the autocorrelations of waiting times is Lindley's relation for the waiting times of two successive customers in a general GI/G/1 system with service in order of arrival:

$$W_{k+1} = \max\{0, W_k + B_k - A_{k+1}\}, \quad k = 1, 2, \ldots ; \tag{4}$$