The Robust Shortest Path Problem by Means of Robust Linear Optimization*

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Abstract. We investigate the robust shortest path problem using the robust linear optimization methodology as proposed by Ben-Tal and Nemirovski. We discuss two types of uncertainty, namely, box uncertainty and ellipsoidal uncertainty. In case of box uncertainty, the robust counterpart is simple. It is a shortest path problem with the original arc lengths replaced by their upper bounds. When dealing with ellipsoidal uncertainty, we obtain a conic quadratic optimization problem with binary variables. We present an example to show that a subpath of a robust shortest path is not necessarily a robust shortest path.

1 Introduction

The shortest path problem (SPP) is one of the simplest and most well-studied combinatorial optimization problems. For a given network $G = (\mathcal{V}, \mathcal{A})$ with node set $\mathcal{V}$ and arc set $\mathcal{A}$, an instance of the SPP is finding a directed path from a single node s to a single node t with a minimal total length, with respect to a given length function $\mathbf{c} : \mathcal{A} \to \mathbb{R}_+$. A binary optimization model for the shortest path problem is

$$\min\{c^T x \ : \ Ax = b, x \in \{0,1\}^{\mathcal{A}}\}, \tag{1}$$

where A is the node-arc incident matrix, b is the vector in $\mathbf{R}^{\mathcal{V}}$ with entries $b_s = 1$, $b_t = -1$ and $b_j = 0$ for all $j \in \mathcal{V} \setminus \{s,t\}$, and c is the vector of arc lengths. It is well known that the constraint matrix A is a totally unimodular and hence we can equally consider the linear optimization problem

$$\min\{c^T x \ : \ Ax = b, x \in [0,1]^{\mathcal{A}}\}.$$

Recently, some authors considered the SPP for the case where the arc lengths are not certain. The aim is then to find a so called robust shortest

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path, i.e., a path that behaves relatively stable under all possible realizations of the uncertain arc lengths. For instance, when a communication network is used to send packages from a source to a sink, an uncertain delay time can occur. Treating the uncertain delay time between two nodes as an uncertain arc length, we obtain an uncertain SPP.

Karasan et al. [4] and Bertsimas et al. [5] considered the uncertain shortest path problem. They model data uncertainty by assuming that each arc length belongs to an interval. We will refer to this case as box uncertainty. Karasan et al. define the robust path as a path with minimum deviation, i.e., the minimal difference between the worst case path length and the shortest path length for all possible realizations. A variant of this approach is proposed by Bertsimas et al. who assume that the uncertainty occurs only on a fixed number of the arcs.

In this paper we apply the Robust Linear Optimization (RLO) methodology as proposed by Ben-Tal and Nemirovski (see [1–3]). This methodology applies to general LO problem. We apply it in this paper to the SPP.

The paper is organized as follows. Section 2 briefly introduces the theory of Robust Linear Optimization. Section 3 is devoted to the RSPP and gives an illustrative example. Conclusions can be found in Section 4.

2 Robust Linear Optimization

We briefly recall some definitions and some main results from [3]. Consider a linear optimization problem

$$\min_x \{c^T x \; : \; Ax \geq b\}, \tag{$\mathcal{LO}$}$$

and let $\mathcal{U}$ be the set of all possible realizations of (A, b, c). So the set $\mathcal{U}$ models the uncertainty in the data, A, b and c, of $(\mathcal{LO})$. We call $\mathcal{U}$ the uncertainty set. As a consequence, we have a whole family of $\mathcal{LO}$ problems, for each $(c, A, b) \in \mathcal{U}$, one $\mathcal{LO}$ problem. This family is given by

$$\left\{ \min_x \{c^T x \; : \; Ax \geq b\} \; : \; (c, A, b) \in \mathcal{U} \right\}. \tag{2}$$

Instead we consider the so-called robust counterpart of (2), namely

$$\min \left\{ \ell \; : \; \ell \geq c^T x, Ax - b \geq 0, \forall (c, A, b) \in \mathcal{U} \right\}. \tag{3}$$

The formulation (3) is a linear optimization problem with usually infinitely many constraints, depending on the uncertainty set $\mathcal{U}$. Hence, in general this problem may be very hard to solve. Special cases for $\mathcal{U}$ make (3) computational tractable. In [1], it has been shown that the robust counterpart (3) is equivalent to an explicit computationally tractable problem provided that $\mathcal{U}$ has a simple structure. This becomes clear from the following theorem which presents three examples of computational tractable uncertainty sets.