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Relations, Schedules, and Objective Functions

When allocating scarce resources over time we have to define precedence relationships among the activities of the project. Those precedence relationships establish a binary relation in the activity set of the project. Together with the original temporal constraints, the binary relation gives rise to a preorder in the activity set. Depending on the type of basic project scheduling problem given and the specific objective function to be minimized, different types of preorders have to be investigated. In this chapter we review and extend a classification of schedules and objective functions that has been proposed by Neumann et al. (2000). The classification is based on two basic representations of the feasible region of project scheduling problems as unions of relation-induced polytopes. The purpose of the classification is to provide, for each class of objective functions, a finite set of candidates for optimal schedules that are characterized as specific points of the relation-induced polytopes such as minimal points, local minimizers of the objective function, or vertices.

2.1 Resource Constraints and Feasible Relations

Before we discuss the relationship between resource constraints and certain relations in the set of real activities or events, respectively, we first review some basic terminology.

Definition 2.1 (Binary relation, preorder, and strict order). *A binary relation ρ in (ground) set X is a set of pairs $(x, y) \in X \times X$. Relation ρ' in X with $\rho' \supseteq \rho$ is termed an extension of ρ. $tr(\rho)$ denotes the transitive hull of relation ρ, i.e., the $\subseteq$-minimal transitive extension of ρ in X. A transitive binary relation θ in set X is termed a preorder in X. Two elements $x, y \in X$ are referred to as comparable in preorder θ if $(x, y) \in \theta$ or $(y, x) \in \theta$, and incomparable, otherwise. θ is a complete preorder if $(i, j) \in \theta$ or $(j, i) \in \theta$ for all $i, j \in X$, $i \neq j$. A set $U \subseteq X$ of pairwise incomparable elements is called an antichain in θ. $Pred^{\theta}(x) = \{y \in X \mid (y, x) \in \theta\}$ is the set of predecessors of*

x in θ. $x \in Y \subseteq X$ is called a maximal element of Y in θ if $(y,x) \in \theta$ implies $(x,y) \in \theta$ for all $y \in Y$, $y \neq x$. An irreflexive preorder is asymmetric and thus represents a strict order. The covering relation $cr(\theta)$ of strict order θ is the $\subseteq$-minimal binary relation ρ in X with $tr(\rho) = \theta$. The precedence graph of strict order θ is the directed graph $G(\theta)$ with node set X and arc set $cr(\theta)$.

When we deal with renewable resources, forbidden sets F are broken up by introducing precedence constraints $S_j \geq S_i + p_i$ between real activities $i, j \in F$. In other words, we construct a strict order θ in the set V^a of real activities where $(i,j) \in \theta$ means that activity j cannot be started before activity i has been completed. In case of cumulative resources, surplus and shortage sets F are broken up by introducing precedence constraints $S_j \geq S_i$ between events $i \in V^e \setminus F$ and events $j \in F$. Thus, by resolving cumulative-resource conflicts we establish a reflexive preorder θ in event set V^e whose elements (i,j) say that event j cannot take place before the occurrence of event i.

The following two types of preorders will be needed when studying precedence relationships between real activities or events that are induced by a given schedule.

Definition 2.2 (Interval order and weak order). *An interval order in set X is a strict order θ in X for which $(w,x),(y,z) \in \theta$ implies $(w,z) \in \theta$ or $(y,x) \in \theta$ for all $w,x,y,z \in X$. A (reflexive) weak order in set X is a complete and reflexive preorder in X.*

2.1.1 Renewable-Resource Constraints

In this subsection we consider irreflexive relations in the set V^a of real activities for the scheduling of projects with renewable resources. We first define the concepts of time-feasible and feasible relations, which go back to the work of Radermacher (1978) and Bartusch et al. (1988). In difference to the treatment of the material by Neumann et al. (2000) and Neumann et al. (2003*b*), Sect. 2.3, we use relations instead of strict orders, which allows of a unifying view on renewable-resource and cumulative-resource constraints.

Definition 2.3 (Time-feasible and feasible relations). *Let ρ be an irreflexive relation in set V^a and let $\mathcal{S}_T(\rho) := \{S \in \mathcal{S}_T \mid S_j \geq S_i + p_i \text{ for all } (i,j) \in \rho\}$ be the set of all time-feasible schedules satisfying the precedence constraints given by ρ. $\mathcal{S}_T(\rho)$ is called the relation polytope of ρ. Relation ρ is termed time-feasible if $\mathcal{S}_T(\rho) \neq \emptyset$. A time-feasible relation ρ with $\mathcal{S}_T(\rho) \subseteq \mathcal{S}$ is called feasible.*

Condition $\mathcal{S}_T(\rho) \neq \emptyset$ means that the precedence constraints from relation ρ do not contradict the prescribed temporal constraints. If $\mathcal{S}_T(\rho) \subseteq \mathcal{S}$, all schedules satisfying those precedence constraints are feasible. If ρ is a feasible relation, then all time-feasible extensions $\rho' \supset \rho$ are feasible as well. A feasible relation ρ represents a solution to the *sequencing problem* of resource