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Relaxation-Based Algorithms

Relaxation-based algorithms for resource-constrained project scheduling with regular or convexifiable objective functions rely on the first basic representation of the set $\mathcal{S}$ of all feasible schedules as a union of relation polytopes. By deleting the resource constraints we obtain the resource relaxation, which coincides with the time-constrained project scheduling problem. The latter problem can be solved efficiently by computing the minimal point ES of set $\mathcal{S}_T$ if f is regular or some local minimizer of the objective function f in set $\mathcal{S}_T$ if f is convexifiable. Clearly, the tractability of the problem is preserved when moving from set $\mathcal{S}_T$ to arbitrary nonempty relation polytopes $\mathcal{S}_T(\rho)$. Starting with the resource relaxation, i.e., with the empty relation, relaxation-based algorithms iteratively put the resource constraints into force by branching over time-feasible extensions ρ' of the respective parent relation ρ. Each relation ρ' defines a collection of precedence constraints that break up some forbidden active set $\mathcal{A}(S,t)$ belonging to a minimizer S of f on search space $\mathcal{P} = \mathcal{S}_T(\rho)$. The branching process is continued until either $\mathcal{S}_T(\rho) = \emptyset$ or the minimizer S of f on $\mathcal{S}_T(\rho)$ is feasible. The latter condition is necessarily satisfied as soon as relation ρ is feasible. Note, however, that schedule S may be feasible even before ρ has been extended to a feasible relation. When dealing with regular objective functions, the ordinary precedence constraints given by relations ρ may be replaced by disjunctive precedence constraints (cf. Subsections 1.2.3 and 1.3.3). Since a disjunctive precedence constraint corresponds to the disjunction of several ordinary precedence constraints, branching is then performed over sets of relations and consequently, the search spaces $\mathcal{P}$ on which f is to be minimized represent unions of relation polytopes.

From now on we assume that the project under consideration comprises renewable and cumulative resources, where the renewable resources are used by real activities $i \in V^a$ and the cumulative resources are depleted and replenished by events $i \in V^e$. Accordingly, for given schedule S the active sets

$$\mathcal{A}(S,t) := \{i \in V^a \mid S_i \leq t < S_i + p_i\} \cup \{i \in V^e \mid S_i \leq t\}$$

at times t contain both real activities and events, and resource-feasible schedules satisfy both the renewable-resource constraints (1.7) and the cumulative-resource constraints (1.20). The set of all feasible schedules is now $\mathcal{S} = \mathcal{S}_T \cap \mathcal{S}_R \cap \mathcal{S}_C$. As a straightforward extension of the definitions from Subsections 2.1.1 and 2.1.2, we say that a relation ρ *in set* V is time-feasible if $\mathcal{S}_T(\rho) \neq \emptyset$ and is feasible if $\emptyset \neq \mathcal{S}_T(\rho) \subseteq \mathcal{S}$. It is easily seen that first, relation ρ is again time-feasible precisely if relation network $N(\rho)$ does not contain any cycle of positive length and that second, a time-feasible relation ρ is feasible exactly if both induced sub-relations $\rho \cap (V^a \times V^a)$ and $\rho \cap (V^e \times V^e)$ are feasible in the sense of Definitions 2.3 and 2.17. As a consequence of the latter statement, the feasibility of a time-feasible relation ρ in set V can be verified by sequentially applying the network flow techniques discussed in Subsections 2.1.1 and 2.1.2 to the respective sub-relations.

The *resource-constrained project scheduling problem* to be dealt with reads as follows:

$$\boxed{\left.\begin{array}{ll} \text{Minimize} & f(S) \\ \text{subject to} & S \in \mathcal{S}_T \cap \mathcal{S}_R \cap \mathcal{S}_C \end{array}\right\} \text{(P)}}$$

where f is some regular or convexifiable objective function. In Section 3.1 we treat the case of regular objective functions. Section 3.2 is devoted to convexifiable objective functions.

3.1 Regular Objective Functions

We first develop an enumeration scheme based on the concept of disjunctive precedence constraints that either generates a set of candidate schedules containing an optimal schedule or proves that there is no feasible schedule for the project under consideration. We are then concerned with the relaxation to be solved at each enumeration node. The latter problem amounts to minimizing a regular objective function subject to temporal and disjunctive precedence constraints. Next, we discuss the extension of the enumeration scheme to a branch-and-bound algorithm and review alternative solution procedures for resource-constrained project scheduling with regular objective functions.

3.1.1 Enumeration Scheme

In this subsection we are concerned with an enumeration scheme for problem (P) with regular objective function f which forms the basis of branch-and-bound procedures by Schwindt (1998*a*) and Neumann and Schwindt (2002) for solving the project duration problem with renewable or cumulative resources, respectively. Consider an optimal solution S to the time-constrained project scheduling problem (1.2) with a regular objective function f, e.g., $S = ES = \min \mathcal{S}_T$. If S satisfies the renewable-resource constraints (1.7) and