Chapter 1

Odds and ends

One purpose of this chapter is to standardize some terminology and notation. In particular, Definition 1.1 defines what we mean by the term “function space,” and Section 1.4 introduces a number of kinds of binary relations. We also use this chapter to present some useful odds and ends that should be a part of everyone’s mathematical tool kit, but which don’t conveniently fit anywhere else. We introduce correspondences and the notion of the evaluation duality. Our presentation is informal and we do not prove many of our claims. We also feel free to get ahead of ourselves and refer to definitions and examples that appear much later on.

We do prove a few theorems including Szpilrajn’s Extension Theorem 1.9 for partial preorders, the existence of a Hamel basis (Theorem 1.8), and the Knaster–Tarski Fixed Point Theorem 1.10. These are presented as applications of Zorn’s Lemma 1.7. Example 1.4 uses a standard cardinality argument to show that the lexicographic order cannot be represented by a numerical function.

We also try to present the flavor of the subtleties of modern set theory without actually proving the results. We do however prove Cantor’s Diagonal Theorem 1.5 and describe Russell’s Paradox. We mention some of the more esoteric aspects of the Axiom of Choice in Section 1.11 in order to convince you that you really do want to put up with it, and all it entails, such as non-measurable sets (Corollary 10.42). We also introduce the ordinals in Section 1.13.

1.1 Numbers

Leopold Kronecker is alleged to have remarked that, “God made the integers, all the rest is the work of man.”[1] The **natural numbers** are $1, 2, 3, \ldots$, etc., and the set of natural numbers is denoted $\mathbb{N}$. (Some authors consider zero to be a natural number as well, and there are times we may do likewise.) We do not attempt to develop a construction of the real numbers, or even the natural numbers here. A very readable development may be found in E. Landau [221] or C. D. Aliprantis and O. Burkinshaw [13, Chapter 1].

[1] According to E. T. Bell [36, p. 477].

We use the symbol $\mathbb{R}$ to denote the set of real numbers, and may refer to the set of real numbers as the **real line**. We use the standard symbols $\mathbb{Z}$ for the integers, and $\mathbb{Q}$ for the rational numbers. We take for granted many of the elementary properties of the real numbers. For instance: Between any two distinct real numbers there are both a rational number and an irrational number. Any nonempty bounded set of real numbers has both an infimum and a supremum. Any nonempty set of nonnegative integers has a least element.

We have occasion to use the **extended real number** system $\mathbb{R}^*$. This is the set of real numbers together with the entities ∞ (infinity) and $-\infty$ (negative infinity). These have the property that $-\infty < r < \infty$ for any real number $r \in \mathbb{R}$. They also satisfy the following arithmetic conventions:

$$\begin{gathered} r + \infty = \infty \quad \text{and} \quad r - \infty = -\infty; \\ \infty \cdot r = \infty \text{ if } r > 0 \quad \text{and} \quad \infty \cdot r = -\infty \text{ if } r < 0; \\ \infty \cdot 0 = 0; \end{gathered}$$

for any real r. The combination $\infty - \infty$ of symbols has no meaning. The symbols ∞ and $-\infty$ are not really meant to be used for arithmetic, they are only used to avoid awkward expressions involving infima and suprema.[2]

1.2 Sets

Informally, a set is a collection of objects. In most versions of set theory, these objects are themselves sets. Even numbers are viewed as sets. We employ the following commonly used set theoretic notation. We expect that this is familiar material, and only mention it to make sure we are all using the same notation. For variety's sake, we may use the term **family** or **collection** in place of the term **set**. The expression $x \in A$ means that x **belongs** to the set A, and $x \notin A$ means that it does not. We may also say that x is a **member** of A, a **point** in A, or an **element** of A, or that A **contains** x if $x \in A$. Two sets are equal if they have the same members. The symbol $\varnothing$ denotes the **empty set**, the set with no members. The expression $X \setminus A$ denotes the **complement** of A in X, that is, the set $\{x \in X : x \notin A\}$. When the reference set X is understood, we may simply write A^c.

The symbols $A \subset B$ or $B \supset A$ mean that the set A is a **subset** of the set B or B is a **superset** of A, that is, $x \in A$ implies $x \in B$. We also say in this case that B

[2] Do not confuse the extended reals with "nonstandard" models of the real numbers. Nonstandard models of the real numbers contain *infinitesimals* (positive numbers that are smaller than every standard positive real number) and *infinitely large* numbers (numbers that are larger than every standard real number), yet nevertheless obey all the rules of real arithmetic (in an appropriately formulated language). See, for instance, R. F. Hoskins [169], A. E. Hurd and P. A. Loeb [173], or K. D. Stroyan and W. A. J. Luxemburg [326] for a good introduction to nonstandard analysis.