Chapter 10

Charges and measures

In Chapter 4 we introduced the concept of σ-algebra of sets to capture the properties of events in probability theory. We used the traditional terminology of referring to sets belonging to such a σ-algebra as *measurable sets*. While we have good pedagogical reasons for introducing the material in this order, it is not obvious what a σ-algebra has to do with measurement of anything. In this chapter we hope to remedy this omission. Historically, mathematicians were interested in generalizing the notions of length, area, and volume. The most useful generalization of these concepts is the notion of a *measure*. In its abstract form a measure is a set function with additivity properties that reflect the properties of length, area, and volume. A *set function* is an extended real function defined on a collection of subsets of an underlying *measurable space*. (We also impose the restriction that a measure assumes at most one of the values ∞ and $-\infty$.) In this chapter we consider set functions that have some of the properties ascribed to area. The main property is *additivity*. The area of two regions that do not overlap is the sum of their areas. A *charge* is any nonnegative set function that is additive in this sense. A *measure* is a charge that is countably additive. That is, the area of a sequence of disjoint regions is the infinite series of their areas. A *probability measure* is a measure that assigns measure one to the entire space. Charges and measures are intimately entwined with *integration*, which we take up in Chapter 11. But here we study them in their own right.

The reason we are interested in charges and measures is that in probability theory and economics, the underlying measurable space has a natural interpretation in terms of states of the world, or in some economic models, as the space of attributes of consumers and/or commodities. See, for instance, M. Berliant [38], W. Hildenbrand [158], L. E. Jones [187, 188, 189] or A. Mas-Colell [241] for a representative sample of this literature. When the underlying measurable space has an interpretation, the set functions also have natural interpretations, such as probability, population, or resource endowments. Thus measures are natural ways to describe the parameters of our models.

On the other hand, due to the Riesz Representation Theorems (see Chapter 14), measure theory can be approached as a branch of the theory of positive operators on Banach lattices, and indeed this approach is often adopted by mathematicians

interested more in the theory than its interpretation. The Radon–Nikodym Theorem 13.18 and the Kakutani Representation Theorem 9.33 show that spaces of measures play a fundamental role in the theory of Banach lattices.

There are too many treatises on measure theory and integration to mention any significant fraction of them. Halmos [148] is a classic. Aliprantis and Burkinshaw [13], Royden [290], and Rudin [291] provide very readable introductions to the Lebesgue measure and its applications. Billingsley [43], Doob [99], and Dudley [104] elaborate on the role of measure theory in the theory of probability. Neveu [261] contains a number of results that do not seem to appear elsewhere. Luxemburg [233] has a very nice brief treatment of (finitely additive) charges, while Bhaskara Rao and Bhaskara Rao [41] present a detailed analysis of them.

Here is a guide to the main points of this chapter. As we said above, much of the interest in measures stems from interest in integration. The modern Lebesgue–Daniell approach to integration differs from the ancient Archimedes–Riemann approach in the following way. The Riemann integral is calculated by dividing the domain of a function into manageable regions (intervals), approximating the value of the function on each region, summing, and passing to the limit as the size of the regions goes to zero. The Lebesgue approach starts by partitioning the range of the function into small pieces, finding regions in the domain on which the function is approximately constant (these regions may be quite complicated), measuring the size of these regions, summing and passing to the limit as the size of pieces in the range goes to zero. In order to pursue this approach, we need to be able to measure complicated pieces of the domain. Furthermore, when we look for places where the value of a function is nearly constant, we are looking at the inverse image of a small interval. Thus we want the collection of sets that we can measure to include the inverse image of every real interval.

At this point you may ask, *why can't we measure all subsets of the domain?* The answer to this question is quite subtle and takes us into the realm of axiomatic set theory, and the **Problem of Measure**. The Problem of Measure is this: Given any set, is there a probability measure defined on its power set such that every singleton has measure zero?[1] Clearly the answer to this question can only depend on the cardinality of the set. The cardinality of a set is said to be a **measurable cardinal** if the answer to this question is yes. If X is countable, then countable additivity implies that no such probability measure exists. But what if X is uncountable? This is where the set theory comes in. The **Continuum Hypothesis** asserts that $\mathfrak{c}$, the cardinality of the continuum (that is, the cardinality of $[0, 1]$), is the smallest uncountable cardinal. So the Continuum Hypothesis asserts that any uncountable set must have cardinality at least $\mathfrak{c}$. The Continuum Hypothesis, like the Axiom of Choice, is one of those agnostic axioms of ZF set theory—you may assume it without creating contradictions, yet you cannot prove it, even using the

[1] The Ultrafilter Theorem 2.19 implies that for any infinite set there is a probability charge that assigns mass zero to each point. Every free ultrafilter defines a charge assuming only the values zero and one; see Lemma 16.35