Chapter 11

Integrals

In modern mathematics the process of computing areas and volumes is called *integration*. The computation of areas of curved geometrical figures originated about 2,300 years ago with the introduction by the Greek mathematician Eudoxus (ca. 365–300 B.C.E.) of the celebrated "method of exhaustion." This method also introduced the modern concept of limit. In the method of exhaustion, a convex figure is approximated by inscribed (or circumscribed) polygons—whose areas can be calculated—and then the number of vertexes of the inscribed polygons is increased until the convex region has been "exhausted." That is, the area of the convex region is computed as the limit of the areas of the inscribed polygons. Archimedes (287–212 B.C.E.) used the method of exhaustion to calculate the area of a circle and the volume of a sphere, as well as the areas and volumes of several other geometrical figures and solids. The method of exhaustion is, in fact, at the heart of all modern integration techniques.

The method of exhaustion, along with most ancient mathematics, was forgotten for almost 2,000 years until the invention of calculus by I. Newton (1642–1727) and G. Leibniz (1646–1716). The theory of integration then developed rapidly. A.-L. Cauchy (1789–1857) and G. F. B. Riemann (1826–1866) were among the first to present axiomatic abstract foundations of integration.

In the modern abstract approach to integration theory, we usually start with a *measure space* (X, Σ, μ) and the associated Riesz space L of all Σ-*step functions*. The Σ-step functions are the analogues of the inscribed (or circumscribed) polygons. If $\varphi = \sum_{i=1}^{n} a_i \chi_{A_i}$ is a Σ-step function, then the integral of φ is defined as a weighted sum of its values, the weights being the measures of the sets on which φ assumes those values. That is,

$$\int \varphi \, d\mu = \sum_{i=1}^{n} a_i \mu(A_i).$$

The integration problem now consists of finding larger classes of functions for which the integral can be defined in such a way that it preserves the fundamental properties of area and volume. This means that on the larger class (vector space) of functions the integral must remain a positive linear functional possessing a continuity property that captures the exhaustion property of Eudoxus. The measure-

theoretic approach to integration was developed through the work of H. Lebesgue (1875–1941), C. Carathéodory (1873–1950), and P. J. Daniell (1889–1946). Their ideas and approach are present throughout this chapter.

An even more abstract approach to integration is as a positive operator on a Banach lattice. D. H. Fremlin [128] and K. Jacobs [177] are exemplars of this approach. In Chapter 14 we present typical results along these lines.

11.1 The integral of a step function

In this section, $\mathcal{A}$ is an algebra of subsets of a set X and $\mu\colon \mathcal{A} \to [0, \infty]$ denotes a charge. That is, μ is a nonnegative finitely additive set function defined on $\mathcal{A}$.

11.1 Definition *A simple function $\varphi\colon X \to \mathbb{R}$ is a **μ-step function** (or simply a **step function** when the charge μ is well understood) if its standard representation $\varphi = \sum_{i=1}^{n} a_i \chi_{A_i}$ satisfies $\mu(A_i) < \infty$ for each i.*[1]

*A **representation** for a μ-step function φ is any expression of the form $\varphi = \sum_{j=1}^{m} b_j \chi_{B_j}$, where $B_j \in \mathcal{A}$ and $\mu(B_j) < \infty$ for each j.*

In other words, a simple function is a μ-step function if and only if the function vanishes outside of a set in $\mathcal{A}$ of finite measure. So if L denotes the collection of all μ-step functions, then a repetition of the proof of Lemma 4.34 yields the following.

11.2 Lemma *The collection L of all μ-step functions is a Riesz space and, in fact, a function space and an algebra.*

Any satisfactory theory of integration has to treat step functions in the obvious way. That is, the integral of a step function should be a weighted sum of its values, the weights being the measures of the sets on which it assumes those values. Precisely, we have the following definition.

11.3 Definition *Let μ be a charge on an algebra of subsets of a set X, and let $\varphi\colon X \to \mathbb{R}$ be a step function having the standard representation $\varphi = \sum_{i=1}^{n} a_i \chi_{A_i}$. The **integral** of φ (with respect to μ) is defined by*

$$\int \varphi \, d\mu = \sum_{i=1}^{n} a_i \mu(A_i).$$

[1] This terminology is useful, but a little bit eccentric. Many authors reserve the term "step function" for a simple function whose domain is a closed interval of the real line and has a representation in terms of indicators of intervals. It is handy though to have a term to indicate a simple function that is nonzero on a set of finite measure.