Chapter 12

Measures and topology

Chapter 10 dealt with measures and charges defined on abstract semirings or algebras of sets. In applications there is often a natural topological or metric structure on the underlying measure space. By combining topological and set theoretic notions it is possible to develop a richer and more useful theory. Some of these connections between measure theory and topology are discussed in this chapter.

One of the most useful notions involving the topological structure is *tightness*, which asserts that the measure of any measurable set can be approximated by the measure of an included compact set. Indeed if a charge (which need only be finitely additive) on the Borel sets of a Hausdorff space is tight, then it is automatically countably additive (Theorem 12.4). A somewhat stronger condition than tightness is regularity. A measure is *regular* if every compact set has finite measure, it is tight, and in addition, the measure of every set can be approximated by the measure of open sets that include it. Every finite Borel measure on a Polish space is regular (Theorem 12.7). This is not generally true for non-Polish spaces. Example 12.9 is a classic example of a non-regular Borel probability measure on a compact Hausdorff space.

There are other nice properties of Borel measures on Polish spaces. One of these is that in this case (as well as few other cases) every finite measure has a well defined *support*, or minimal closed set of full measure (Theorem 12.14). Example 12.15 shows that in general, even on a compact Hausdorff space, a Borel measure need not have a support. Lusin's Theorem 12.8 shows that when the domain is a Polish space, a Borel measurable function is continuous when restricted to a compact set whose complement has arbitrarily small positive measure. In addition, Theorem 12.22 shows the existence of *nonatomic* regular Borel measures on uncountable Polish spaces.

Section 12.5 discusses analytic subsets of Polish spaces, which are the continuous images of the Baire space $\mathcal{N}$. Every Borel set is analytic (Theorem 12.25), but not vice versa (Example 12.33). However, every analytic set is *universally measurable* (Theorem 12.41), that is, μ-measurable for any Borel probability measure μ. Analytic sets occur naturally in connection with measurable correspondences, see Theorem 18.21. They arise naturally in the study of stochastic processes (see, e.g., Dellacherie [86, 87]), dynamic programming (see, e.g., Bertsekas

and Shreve [39]), and also in the theory of games with incomplete information (see, e.g., Stinchcombe and White [320]). They also appear prominently in Chapter 18 on measurable correspondences below.

Finally we prove some interesting facts about functions between Polish spaces. Theorem 12.28 asserts that a function is Borel measurable if and only if its graph is a Borel set. Theorem 12.29 says that the one-to-one image of a Borel set under a Borel measurable function is a Borel set.

The classic reference for measures on topological spaces is Halmos [148]. Some of this material is covered in standard analysis texts, such as, Aliprantis and Burkinshaw [13], Royden [290], and Rudin [291]. Billingsley [42], Neveu [261], Parthasarathy [271], and Pollard [280] concentrate on applications to probability and stochastic processes. Choquet [76] has an excellent treatment of the topological properties of spaces of Radon measures. The material on Borel functions and analytic sets derives from Kechris [196], Kuratowski [218], Lusin [231], and Parthasarathy [271]. There is also an excellent monograph by Srivastava [319].

12.1 Borel measures and regularity

In this section X is a topological space. As before, the σ-algebra of all Borel sets of X is denoted $\mathcal{B}$, or $\mathcal{B}_X$ or $\mathcal{B}_{orel}$. Similarly, the σ-algebra of all Baire sets is denoted $\mathcal{B}_{aire}$. The symbol $\mathcal{A}_X$ denotes the algebra generated by the open sets.

12.1 Definition *A **(signed) Borel measure** is simply a (signed) measure defined on the Borel sets of a topological space.*[1]

*Similarly, a **(signed) Baire measure** is any (signed) measure defined on the σ-algebra $\mathcal{B}_{aire}$ of Baire sets of a topological space.*

*A **(signed) Borel charge** is a (signed) charge that is defined either on the algebra $\mathcal{A}_X$ or σ-algebra $\mathcal{B}_X$ generated by the open sets.*

While we are more interested in charges and measures than their signed counterparts, we make the following general definitions.

12.2 Definition *Here $\mathcal{A}$ may stand for $\mathcal{A}_X$, $\mathcal{B}_X$, or $\mathcal{B}_{aire}(X)$. Let μ be a charge or measure on $\mathcal{A}$. The charge or measure μ is:*

- ***outer regular** if for every set A in $\mathcal{A}$,*

$$\mu(A) = \inf\{\mu(V) : V \in \mathcal{A},\ V \textit{ open, and } A \subset V\}.$$

[1] Recall that in Chapter 10 we required a Borel measure to assign finite measure to every compact set. Most authors make this definition. However, for the purposes of this chapter, we do not make that requirement. We do require that what we call a regular Borel measure assign finite measure to every compact set.