Chapter 13

L_p-spaces

In this chapter, we introduce the classical L_p-spaces and study their basic properties. Recall that for a measure space (X, Σ, μ), two measurable real functions f and g on X are *equivalent* if they agree μ-almost everywhere. For $0 < p < \infty$, the p-norm of f is defined by

$$\|f\|_p = \Big(\int |f|^p \, d\mu \Big)^{\frac{1}{p}},$$

where we allow the integral to be infinite. Note that this integral depends only on the equivalence class of f. The space $L_p(\mu)$ is the collection of equivalence classes of measurable functions f for which the p-norm is finite. The space $L_\infty(\mu)$ comprises the equivalence classes of essentially bounded measurable functions, while $L_0(\mu)$ is the collection of the equivalence classes of measurable functions. With the pointwise algebraic and lattice operations all the L_p-spaces are order complete Riesz spaces. In fact, for $1 \leqslant p \leqslant \infty$, the $L_p(\mu)$-spaces are all Banach lattices (Theorem 13.5). For $0 \leqslant p < 1$ the $L_p(\mu)$-spaces are not Banach lattices, indeed they are not locally convex topological vector spaces, but they are nevertheless Fréchet lattices (Theorem 13.31). Theorem 13.11 proves the remarkable result that for a probability measure on the σ-algebra Σ, the Banach sublattices of $L_p(\Sigma)$ that contain the constant function $\mathbf{1}$ are exactly the Banach sublattices of the form $L_p(\mathcal{A})$ for some σ-subalgebra $\mathcal{A}$ of Σ.

The duals of the L_p spaces have an interesting representation—they are also L_p-spaces. Theorem 13.26, due to F. Riesz, asserts that if $1 < p, q < \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$, then the norm dual of $L_p(\mu)$ can be identified with $L_q(\mu)$ (and vice-versa by symmetry).

Besides norm convergence in L_p-spaces, there is another natural notion of convergence, convergence in measure. A sequence of measurable functions $\{f_n\}$ *converges in measure* to a measurable function f if for each $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \mu(\{x \in X : |f_n(x) - f(x)| \geqslant \varepsilon\}) = 0.$$

Convergence in measure gives rise to the smallest Hausdorff locally solid topology on an L_p-space, and it is seldom locally convex. As a matter of fact, we establish

that for finite nonatomic measure spaces, the topological dual of any $L_p(\mu)$-space with the topology of convergence in measure is trivial (Theorem 13.41).

We discuss several other topics related to L_p-spaces. For instance, we pay special attention to the Radon–Nikodym Theorem 13.18 and its applications. In particular, we use the Radon–Nikodym Theorem to prove Lyapunov's Convexity Theorem 13.33, which states that the range of finite-dimensional vector of non-atomic finite measures is a compact convex set. The chapter ends with a brief study of the extremely useful "Change of Variables" formulas.

13.1 L_p-norms

In this section (X, Σ, μ) will always be a measure space. A μ-measurable function $f\colon X \to \mathbb{R}$ is ***p*-integrable** (for $0 < p < \infty$) if $|f|^p$ is an integrable function. The set of p-integrable functions is denoted $L_p(\mu)$. Actually it is customary to identify functions that are equal almost everywhere. So $L_p(\mu)$ consists of equivalence classes rather than functions. We do this so the formulas below define norms and not just seminorms.

For $0 < p < \infty$, the set $L_p(\mu)$ is actually a vector space under the pointwise operations. It is clearly closed under scalar multiplication. To see that the sum of two p-integrable functions is also p-integrable, observe that for any pair a, b of real numbers, if $|a| \leqslant |b|$, then $|a+b|^p \leqslant (|a|+|b|)^p \leqslant (2|b|)^p \leqslant 2^p\left(|a|^p + |b|^p\right)$. This implies that

$$|f+g|^p \leqslant 2^p\left(|f|^p + |g|^p\right),$$

so $f+g$ is p-integrable if both f and g are.

If $f \in L_p(\mu)$, then the ***L*$_p$-norm** of f is defined by

$$\|f\|_p = \left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}}.$$

The $\|\cdot\|_\infty$-norm (or the **essential sup norm**) of a μ-measurable function $f\colon X \to \mathbb{R}$ is defined by

$$\|f\|_\infty = \inf\{M > 0 : |f(x)| \leqslant M \text{ for } \mu\text{-almost all } x\},$$

where the convention $\inf \varnothing = \infty$ applies. The collection of all equivalence classes of measurable functions f with $\|f\|_\infty < \infty$ is denoted $L_\infty(\mu)$. We let $L_0(\mu)$ denote the set of equivalence classes of measurable functions. In all cases, $L_p(\mu)$ is a vector space. The next result justifies the symbol $\|\cdot\|_\infty$ used to designate the essential sup norm.

13.1 Lemma *If (X, Σ, μ) is a finite measure space and $f \in L_\infty(\mu)$, then*

$$\lim_{p \to \infty} \|f\|_p = \|f\|_\infty.$$