Chapter 14

Riesz Representation Theorems

In this chapter we discuss a well-known family of theorems, known collectively as the Riesz Representation Theorems, that assert that positive linear functionals on the classical normed Riesz space $C(X)$ of continuous real functions on X can be represented as integrals with respect to Borel measures. To make sure everything is integrable, we restrict attention either to continuous functions with compact support, $C_c(X)$, and measures that are finite on compact sets, or to finite measures and bounded continuous functions, $C_b(X)$. We also consider positive functionals on the spaces of bounded measurable real functions $B_b(X)$.

Theorem 14.9 asserts that a positive linear functional on $C_b(X)$, the space of bounded continuous real functions on X, where X is a normal Hausdorff space, has a representation as the integral with respect to a unique outer regular charge on the algebra generated by the open sets. A charge is *outer regular* if every set can be approximated (in measure) from the outside by open sets. Since $C_b(X)$ is a Banach lattice, every positive linear functional is norm continuous. Theorem 14.10 shows that the space of outer regular charges with the usual lattice operations is lattice isometric to the norm dual of $C_b(X)$.

Theorem 14.12 asserts that a positive linear functional on $C_c(X)$, the space of continuous real functions on X with compact support, where X is a locally compact Hausdorff space, has a representation as the integral with respect to a unique regular Borel measure. Indeed, positive linear functionals on $C_c(X)$ are often called *Radon measures*. A Borel measure is *regular* if it is outer regular and *tight*, meaning every Borel set can be approximated in measure from inside by compact sets. Theorem 14.14 shows that the AL-space of regular Borel measures is lattice isometric to the norm dual of $C_c(X)$. Note that every Borel measure defines a continuous linear functional on $C_c(X)$. However it is possible for two distinct Borel measures on X to define the same linear functional on $C_c(X)$ (Example 14.13). Consequently, $C_c(X)$ does not separate the points in the space of Borel measures. This means that the pairing of $C_c(X)$ with the space of Borel measures is not a dual pair. This problem is cured by restricting attention to regular Borel measures.

There are many versions of these theorems that appear in the literature, and the relations among them are not always clear. For instance, one version states

that a positive linear functional on $C(X)$, where X is a compact Hausdorff space, has a representation as the integral with respect to a unique finite Baire measure. The way this result relates to Theorem 14.12 is this. We know that we only need Baire measures to be able to integrate functions in $C_c(X)$. In the locally compact case, every positive functional has a representation in terms of a Baire measure. This representation may not be unique in the space of Baire measures. The way we get a unique measure is by requiring it to have a regular extension to the Borel σ-algebra. In the smaller class of regular Borel measures the representation is unique. In the special case where X is compact, the representation is already unique in the class of Baire measures. There is still a representation as a regular Borel measure, but this may not be stated.

We also prove (Corollary 14.15) that when X is compact and metrizable, and so a special case of both locally compact and normal space, a positive linear functional on $C(X) = C_b(X) = C_c(X)$ has a representation as the integral with respect to a unique finite regular Borel measure. This is reconciled with Theorem 14.9 by showing that for compact metrizable spaces, every outer regular charge on the algebra generated by the open sets is the restriction of a unique regular Borel measure.

We also show that every continuous linear functional on the space $B_b(\Sigma)$ of bounded measurable functions on a σ-algebra has a representation as a signed charge (Lemma 14.3). These and other representation theorems are summarized in Table 14.1 on page 499.

Theorem 14.23 characterizes homomorphisms between $C(X)$ spaces, where X is compact, as *composition operators.*

14.1 The AM-space $B_b(\Sigma)$ and its dual

In this section Σ is a σ-algebra of subsets of some fixed set X.

14.1 Definition *The collection of all bounded Σ-measurable real functions defined on X is denoted $B_b(\Sigma)$.*

Recall that if X is a topological space, then for simplicity we write $B_b(X)$ instead of $B_b(\mathcal{B}_X)$. When $B_b(\Sigma)$ is equipped with the sup norm it becomes an AM-space having unit the constant function $\mathbf{1}$. That is:

14.2 Theorem *The Riesz space $B_b(\Sigma)$ equipped with the sup norm is a σ-order complete AM-space with unit* $\mathbf{1}$.

Next we describe the norm dual of the Banach lattice $B_b(\Sigma)$. Recall that since $B_b(\Sigma)$ is an AM-space, its norm dual $B_b'(\Sigma)$ is an AL-space and coincides with its order dual (Theorems 9.11 and 9.27).