Chapter 15

Probability measures

Unless otherwise indicated, in this chapter X is a metrizable topological space, and $\mathcal{P}(X)$ (or simply $\mathcal{P}$) is the set of all probability measures on the Borel sets $\mathcal{B}$ of X. As usual, $C_b(X)$ denotes the Banach lattice of all bounded continuous real functions on X. The reason we focus on probability measures is that the probability measures span the space of all signed measures of bounded variation.

Recall that a **probability measure**, $\mu\colon \mathcal{B} \to [0,1]$ is a measure with $\mu(X) = 1$. We use the phrase "a probability measure on a topological space X" synonymously with "a probability measure on the Borel σ-algebra $\mathcal{B}_X$." The set $\mathcal{P}(X)$ is endowed with the topology $w^* = \sigma(\mathcal{P}(X)C_b(X))$.

In this chapter we study the topological properties of $\mathcal{P}(X)$. First, we characterize w^*-convergence in $\mathcal{P}(X)$ by means of topological properties of the space X. The space X can be viewed as a subset of $\mathcal{P}(X)$ by identifying each $x \in X$ with the point mass δ_x. This identification is an embedding (Theorem 15.8) and in case X is also separable, each point in X is an extreme point of $\mathcal{P}(X)$ (Theorem 15.9). The space $\mathcal{P}(X)$ inherits many of the properties of X. For instance, for a metrizable topological space X, we prove:

1. X is compact if and only if $\mathcal{P}(X)$ is compact.

2. X is separable if and only if $\mathcal{P}(X)$ is separable.

3. X is Polish if and only if $\mathcal{P}(X)$ is Polish.

4. X is a Borel space if and only if $\mathcal{P}(X)$ is a Borel space.

By the definition of the $w^* = \sigma(\mathcal{P}(X), C_b(X))$ topology, for every bounded continuous real function f on X, the bounded real function on $\mathcal{P}(X)$ defined by $\mu \mapsto \int f\,d\mu$ is w^*-continuous. Moreover, we shall see that bounded semicontinuous functions define on X define bounded semicontinuous functions on $\mathcal{P}(X)$ (Theorem 15.5), and when X is separable, bounded measurable functions on X define bounded measurable functions on $\mathcal{P}(X)$ (Theorem15.13).

The chapter ends with a discussion of infinite products and the Kolmogorov Extension Theorem 15.26.

15.1 The weak* topology on $\mathcal{P}(X)$

Recall that $U_d(X)$ (or simply U_d) denotes the set of all bounded d-uniformly continuous real functions on X. The set U_d contains the constant functions, and by Corollary 3.15 it is pointwise dense in $C_b(X)$. Moreover, U_d is closed under addition, scalar multiplication, pointwise multiplication, and the lattice operations. It is also a uniformly closed (that is, a norm-closed) subset of $C_b(X)$. In other words, U_d is a uniformly closed subalgebra of the algebra $C_b(X)$. If X is also compact, then U_d coincides, of course, with $C_b(X) = C(X)$.

Our first result shows that U_d is a total set of linear functionals on the probability measures. That is, U_d separates points.

15.1 Theorem *For two probability measures μ and ν on a metrizable topological space X, the following statements are equivalent.*

1. $\mu = \nu$.
2. $\mu(G) = \nu(G)$ *for all open sets G.*
3. $\mu(F) = \nu(F)$ *for all closed sets F.*
4. $\int f\,d\mu = \int f\,d\nu$ *for all $f \in C_b(X)$.*
5. $\int f\,d\mu = \int f\,d\nu$ *for all $f \in U_d$, where d is any compatible metric.*
6. $\int f\,d\mu = \int f\,d\nu$ *for all $f \in D$, where D is any uniformly dense subset of U_d for some compatible metric d on X.*

Proof: The equivalence of (1), (2) and (3) follows from Corollary 10.11. Also the implications (1) $\implies$ (4) $\implies$ (5) $\implies$ (6) are obviously true. We finish the proof by proving (6) implies (3). So assume that there exists a compatible metric d and a uniformly dense subset D of U_d such that $\int f\,d\mu = \int f\,d\nu$ for all $f \in D$. Now if $f \in U_d$ pick a sequence $\{f_n\} \subset D$ with $\|f_n - f\|_\infty \to 0$. Clearly, $\|f_n\|_\infty < M < \infty$ for all n and some $M > 0$. So by the Lebesgue Dominated Convergence Theorem 11.21, we get

$$\int f\,d\mu = \lim_{n\to\infty} \int f_n\,d\mu = \lim_{n\to\infty} \int f_n\,d\nu = \int f\,d\nu$$

for all $f \in U_d$.

Finally, let F be a closed subset of X. By Corollary 3.14 there exists a sequence $\{f_n\}$ in U_d such that $f_n(x) \downarrow \chi_F(x)$ for all $x \in X$. Therefore, using the Lebesgue Dominated Convergence Theorem 11.21 once more, we see that

$$\mu(F) = \int \chi_F\,d\mu = \lim_{n\to\infty} \int f_n\,d\mu = \lim_{n\to\infty} \int f_n\,d\nu = \int \chi_F\,d\nu = \nu(F),$$

and the proof is finished. ∎