Chapter 16

Spaces of sequences

Among the most important and simplest normed and Banach spaces are the sequence spaces—vector subspaces of the vector space $\mathbb{R}^{\mathbb{N}}$ of all real sequences. In this chapter, we introduce the classical sequence spaces; φ, the space of sequences with only finitely many nonzero terms; c_0, the space of sequences converging to zero; c, the space of all convergent sequences; ℓ_∞, the space of bounded sequences; and ℓ_p $(0 < p < \infty)$, the space of p-absolutely summable sequences. We discuss each of these sequence spaces and investigate its topological and lattice structures, including its duals.

We start with the "universal" sequence space $\mathbb{R}^{\mathbb{N}}$ as a Fréchet lattice. Its topological dual and its order dual coincide, and is φ (Theorem 16.3). The norm dual of φ (with the sup norm) is ℓ_1 (Theorem 16.1). The space c_0 is an AM-space under the sup norm, and its norm dual is also ℓ_1 (Theorem 16.7). The space c is an AM-space with unit. Remarkably the spaces c and c_0 are linearly homeomorphic (Theorem 16.12). The norm dual of c is a little complicated (Theorem 16.14).

The basic properties of the ℓ_p-spaces are discussed with special emphasis on ℓ_1, ℓ_∞, and the symmetric Riesz pair $\langle \ell_\infty, \ell_1 \rangle$. For $1 \leqslant p < \infty$, the norm dual of ℓ_p is ℓ_q where $\frac{1}{p} + \frac{1}{q} = 1$ (Theorem 16.20). The norm dual of ℓ_∞ is $ba(\mathbb{N})$ (Corollary 14.11). We discuss at length the structure of the unit ball in $ba(\mathbb{N})$ in terms of ultrafilters in section 16.8. In particular, we show that ultrafilters on $\mathbb{N}$ correspond to zero-one valued probability charges on $\mathbb{N}$, and that the free ultrafilters are the weak$*$-limit points of the point-mass probability charges.

The sequence spaces can be thought of as the "building blocks" of Banach spaces and Banach lattices. Whether they are *embeddable* in a Banach space or a Banach lattice reflect the topological and order structure of the space. We investigate the embeddings of c_0, ℓ_1, and ℓ_∞ into Banach spaces and Banach lattices.

The mapping from a sequence to its limit is a linear functional on the vector space c of convergent sequences. A Banach–Mazur limit is a linear functional on the space ℓ_∞ of all bounded sequences that is an extension the limit functional. The extension is required to be bounded between the $\liminf$ and $\limsup$. We establish the existence of such Banach–Mazur limits (Theorem 16.47), and use them to prove the existence of invariant measures under *flows* on compact metrizable spaces (Theorem 16.48).

We close the chapter with a short discussion of vector-valued sequence spaces, and discuss the generalization of the ℓ_p-spaces from real-valued sequences to vector-valued sequences. In particular, we show that the norm dual of the ℓ_p-sum is the ℓ_q-sum of the sequence of norm duals (Theorem 16.49).

16.1 The basic sequence spaces

Recall that $\mathbb{N}$ denotes the set of natural numbers $\{1, 2, \ldots\}$. Then $\mathbb{R}^{\mathbb{N}}$ is the vector space of all real sequences, that is, real-valued functions on $\mathbb{N}$. Since $\mathbb{N}$ is naturally a separable metric space under the discrete metric, we can choose to think of sequences as continuous functions on $\mathbb{N}$. A **sequence space** is simply any vector subspace of $\mathbb{R}^{\mathbb{N}}$.

As usual, we may write $x = (x_1, x_2, \ldots)$ to denote an element of $\mathbb{R}^{\mathbb{N}}$. If x is a convergent sequence, then we denote its limit by x_∞. Given a sequence x we define the ***n*-tail** of x by

$$x^{(n)} = (0, \ldots, 0, x_{n+1}, x_{x+2}, \ldots)$$

and the ***n*-head** by

$${}^{(n)}x = (x_1, \ldots, x_n, 0, 0, \ldots).$$

There are some special sequences to which we have occasion to refer, and we assign them special symbols. The constant sequence whose terms are all unity is denoted $\boldsymbol{e}$, that is, $\boldsymbol{e} = (1, 1, \ldots)$. The k^{th} unit coordinate vector is the sequence whose k^{th}-term is one and every other term is zero, denoted $\boldsymbol{e}_k$. In a finite dimensional space the unit coordinate vectors form a basis for the space. This is not true in $\mathbb{R}^{\mathbb{N}}$, because any linear combination of unit coordinate vectors is a sequence with only finitely many nonzero terms.

The vector space $\mathbb{R}^{\mathbb{N}}$ is partially ordered by the pointwise ordering, $x \geqslant y$ in $\mathbb{R}^{\mathbb{N}}$ if $x_n \geqslant y_n$ for each n. You should check that $\mathbb{R}^{\mathbb{N}}$ is an order complete Riesz space. Its lattice operations are given pointwise:

$$x \vee y = (\max\{x_1, y_1\}, \max\{x_2, y_2\}, \ldots)$$

and

$$x \wedge y = (\min\{x_1, y_1\}, \min\{x_2, y_2\}, \ldots).$$

For any pair of sequences $x, y \in \mathbb{R}^{\mathbb{N}}$, we define the **dot product** $\langle x, y \rangle$ by

$$\langle x, y \rangle = \sum_{n=1}^{\infty} x_n y_n,$$

provided that the series is convergent in $\mathbb{R}$. We may sometimes write the dot product as $x \cdot y$.