Chapter 17

Correspondences

A *correspondence* is a set-valued function. That is, a correspondence assigns to each point in one set a set of points in a possibly different set. As such, it can simply be identified with a subset of the Cartesian product of the two sets. It may seem a bit silly to dedicate two chapters to such a topic, but correspondences arise naturally in many applications. For instance, the budget correspondence in economic theory associates the set of affordable consumption plans to each price-income combination; the excess demand correspondence is a useful tool in studying economic equilibria; and the best-reply correspondence is the key to analyzing noncooperative games. The theory of “differential inclusions” deals with set-valued differential equations and plays an important role in control theory.

The biggest difference between functions and correspondences has to do with the definition of an inverse image. The inverse image of a set A under a function f is the set $\{x : f(x) \in A\}$. For a correspondence φ, there are two reasonable generalizations, the *upper inverse* of A, which is $\{x : \varphi(x) \subset A\}$, and the *lower inverse* of A, namely $\{x : \varphi(x) \cap A \neq \varnothing\}$. When φ is singleton-valued, both definitions reduce to the inverse of A treating φ as a function.

Having two distinct notions of the inverse leads to (at least) two definitions of continuity. As a result, the terminology has not been fully standardized. We adopt the following definitions. A correspondence is *upper hemicontinuous* if the upper inverse of any open set is open, and *lower hemicontinuous* if the lower inverse of any open set is open. The Closed Graph Theorem 17.11 for correspondences states that a closed-valued correspondence into a compact Hausdorff space is upper hemicontinuous if and only if its graph is closed. Upper hemicontinuous correspondences with compact values mimic the properties of continuous functions reasonably well. For instance the image of a compact set under such correspondences is compact (Lemma 17.8), and products preserve the property (Theorem 17.28).

One of the most useful results involving correspondences is the Maximum Theorem 17.31. This theorem gives sufficient conditions for the set of solutions of a parametric constrained maximization problem to be upper hemicontinuous, and for the optimal value function to be continuous. This theorem is the key result in control theory, equilibrium theory, and game theory. The essential requirements

are that the constraint correspondence be both upper and lower hemicontinuous in the parameters and that the objective function be continuous.

We also present a useful result (Theorem 17.48) on maximal elements of a possibly intransitive and incomplete binary relation. This result is the dual form of K. Fan's extension (Theorem 17.46) of the KKM Lemma. It is the key to a number of useful fixed point theorems. A *fixed point* of a correspondence φ is a point x satisfying $x \in \varphi(x)$. The noted Kakutani–Fan–Glicksberg Fixed Point Theorem 17.55 asserts that an upper hemicontinuous correspondence with nonempty compact convex values from a compact convex subset of a locally convex Hausdorff space into itself has a fixed point. The Brouwer–Schauder–Tychonoff Fixed Point Theorem 17.56 is a special case of this result. These theorems are the fundamental tools of supply and demand analysis, and of the analysis of noncooperative equilibria in games.

In addition, we present the Michael Selection Theorem 17.66, which asserts that a lower hemicontinuous correspondence with nonempty closed convex values into a Banach space admits a continuous *selector*.

The theory of correspondences was first codified by C. Berge [37]. Many of the results of this chapter may be found in K. C. Border [56] for the special case of Euclidean spaces, and more general results may be found in W. Hildenbrand [158], E. Klein and A. C. Thompson [209], and J. C. Moore [253]. More esoteric works include the monographs by J.-P. Aubin and H. Frankowska [24], C. Castaing and M. Valadier [75], J. E. Jayne and C. A. Rogers [183], and I. Kluvánek and G. Knowles [210]. The book by J.-P. Aubin and A. Cellina [22] is a good reference for the theory of differential inclusions. S. Hu and N. S. Papageorgiou [172] have produced an encyclopedic treatment of the whole area.

17.1 Basic definitions

We start with a formal definition of correspondences and related terms.

17.1 Definition *A **correspondence** φ from a set X to a set Y assigns to each x in X a subset $\varphi(x)$ of Y. We prefer to think of φ as a "multivalued function" from X to Y rather than as a function from X to the power set 2^Y of Y.*

*The terms **multifunction** or **set-valued function** are sometimes used to mean a correspondence. We write $\varphi\colon X \twoheadrightarrow Y$ to distinguish a correspondence from a function from X to Y.*

Let $\varphi\colon X \twoheadrightarrow Y$ be a correspondence. As with functions, we refer to X as the **domain** of φ, and Y as the **range space** (or **codomain**). The **image** of a set $A \subset X$ under φ is the set

$$\varphi(A) = \bigcup_{x \in A} \varphi(x).$$