Chapter 18

Measurable correspondences

Throughout this chapter S denotes a measurable space and X is a topological space (usually metrizable or even Polish). We let Σ denote the σ-algebra of measurable subsets of S, and equip X with its Borel σ-algebra $\mathcal{B}_X$. A special case is where S is a topological space and Σ is its Borel σ-algebra. Of primary interest is whether a correspondence $\varphi\colon S \twoheadrightarrow X$ admits a selector that is measurable. Ideally we want a notion of measurability for correspondences so that any measurable correspondence has a measurable selector. Unfortunately this is not straightforward.

An obvious approach is to define measurability in terms of the lower inverse images of Borel sets.[1] It turns out to be extremely restrictive to require the lower inverse image of every Borel subset of X to be measurable, as we demonstrate in Example 18.11. Thus we look at definitions that require either the lower inverse image of closed sets to be measurable or the lower inverse image of open sets to be measurable. For functions it makes no difference, since $f^{-1}(A^c) = [f^{-1}(A)]^c$. This is not true for either the upper or lower inverse of a correspondence, and the two approaches lead to different notions of measurability, unless the correspondence has compact values; see Theorem 18.10. Call a correspondence *measurable* if the lower inverse of every closed set is measurable, and *weakly measurable* if the lower inverse of every open set is measurable. This choice of definitions turns out to lead to some nice results; see for instance, Theorem 18.13.

A weaker notion of measurability for a correspondence is that its graph be a measurable set. By Theorem 12.28 a function between Polish spaces is Borel measurable if and only if its graph is a Borel set. This equivalence fails to be true for correspondences. A weakly measurable correspondence has measurable graph (Theorem 18.6), but a correspondence with measurable graph need not be weakly measurable. There are two ways around this problem. One is to use a larger σ-algebra on S than the Borel σ-algebra. Indeed, the σ-algebra of universally measurable sets seems to be the appropriate one; see Theorem 18.21. If we want to avoid topological restrictions on S, we can assume that Σ is complete for some measure μ; for this approach, see the excellent treatment by E. Klein and A. C. Thompson [209]. Yet another natural notion of measurability for closed-

[1] The use of lower inverses rather than upper inverses is insignificant. Every definition in terms of lower inverses has a corresponding definition in terms of upper inverses.

valued correspondences arises from treating them as functions into the space $\mathfrak{F}$ of nonempty closed sets.

One of the most important results concerning measurable correspondences is the Kuratowski–Ryll-Nardzewski Measurable Selection Theorem 18.13, which asserts that a weakly measurable correspondence with nonempty closed values into a Polish space has a measurable selector. This is applied to prove Filippov's Implicit Function Theorem 18.17 and the Measurable Maximum Theorem 18.19. The Measurable Maximum Theorem is a useful result that gives conditions for the set of solutions of a parametric constrained maximization problem to be measurable as well as for the optimal value function to be measurable.

We also prove a fundamental result (Theorem 18.31) relating the measurability of a correspondence having compact convex values to the measurability of its support functionals. Measurable correspondences can be integrated. The integral is defined to be the set of integrals of selectors from the correspondence. We consider the integration of compact convex-valued correspondences and present the fundamental Theorem 18.37 of V. Strassen.

18.1 Measurability notions

We start with a few natural, but not equivalent, notions of measurability.

18.1 Definition *Let (S, Σ) be a measurable space and X a topological space. We say that a correspondence $\varphi\colon S \twoheadrightarrow X$ is:*

- ***weakly measurable**, if $\varphi^{\ell}(G) \in \Sigma$ for each open subset G of X.*
- ***measurable**, if $\varphi^{\ell}(F) \in \Sigma$ for each closed subset F of X.*
- ***Borel measurable**, if $\varphi^{\ell}(B) \in \Sigma$ for each Borel subset B of X.*

There is nothing special about using the lower inverse rather than the upper inverse in these definitions. For instance, a correspondence φ is weakly measurable if and only if $\varphi^{\mathrm{u}}(F)$ belongs to Σ for every closed set F, since $\varphi^{\mathrm{u}}(F) = [\varphi^{\ell}(F^{\mathrm{c}})]^{\mathrm{c}}$. Note well that weak measurability has nothing to do with weak topologies. Obviously, measurability and weak measurability are weaker conditions than Borel measurability. Also note that we do not require that φ have nonempty values, but observe that if φ is either measurable or weakly measurable, then the set $\{s \in S : \varphi(s) \neq \varnothing\} = \varphi^{\ell}(X)$ is measurable. Thus requiring nonempty values would not affect the measurability of a correspondence.

If φ is singleton-valued, that is, if it defines a function, then measurability, weak measurability, and Borel measurability of φ all coincide with Borel measurability of φ as a function. The main difference between functions and correspondences in terms of inverse images is that taking the inverse images under a function commutes with complementation, union, and intersection. This is not