## Chapter 2

# Topology

We begin with a chapter on what is now known as general topology. Topology is the abstract study of convergence and approximation. We presume that you are familiar with the notion of convergence of a sequence of real numbers, and you may even be familiar with convergence in more general normed or metric spaces. Recall that a sequence $\{x_n\}$ of real numbers converges to a real number $x$ if $\{|x_n - x|\}$ converges to zero. That is, for every $\varepsilon > 0$, there is some $n_0$ such that $|x_n - x| < \varepsilon$ for all $n \geqslant n_0$. In metric spaces, the general notion of the distance between two points (given by the *metric*) plays the role of the absolute difference between real numbers, and the theory of convergence and approximation in metric spaces is not all that different from the theory of convergence and approximation for real numbers. For instance, a sequence $\{x_n\}$ of points in a metric space converges to a point $x$ if the distance $d(x_n, x)$ between $x_n$ and $x$ converges to zero as a sequence of real numbers. That is, if for every $\varepsilon > 0$, there is an $n_0$ such that $d(x_n, x) < \varepsilon$ for all $n \geqslant n_0$. However, metric spaces are inadequate to describe approximation and convergence in more general settings. A very real example of this is given by the notion of pointwise convergence of real functions on the unit interval. It turns out there is no way to define a metric on the space of all real functions on the interval $[0, 1]$ so that a sequence $\{f_n\}$ of functions converges pointwise to a function $f$ if and only if the distance between $f_n$ and $f$ converges to zero. Nevertheless, the notion of pointwise convergence is extremely useful, so it is imperative that a general theory of convergence should include it.

There are many equivalent ways we could develop a general theory of convergence.[1] In some ways, the most natural place to start is with the notion of a *neighborhood* as a primitive concept. A neighborhood of a point $x$ is a collection of points that includes all those "sufficiently close" to $x$. (In metric spaces, "sufficiently close" means within some positive distance $\varepsilon$.) We could define the collection of all neighborhoods and impose axioms on the family of neighborhoods. Instead of this, we start with the concept of an open set. An *open* set is a set that is a neighborhood of all its points. It is easier to impose axioms on

[1] The early development of topology used many different approaches to capture the notion of approximation: closure operations, proximity spaces, $L$-spaces, uniform spaces, etc. Some of these notions were discarded, while others were retained because of their utility.

the family of open sets than it is to impose them directly on neighborhoods. The family of all open sets is called a *topology*, and a set with a topology is called a *topological space*.

Unfortunately for you, a theory of convergence for topological spaces that is adequate to deal with pointwise convergence has a few quirks. Most prominent is the inadequacy of using sequences to describe continuity of functions. A function is continuous if it carries points sufficiently close in the domain to points sufficiently close in the range. For metric spaces, continuity of $f$ is equivalent to the condition that the sequence $\{f(x_n)\}$ converges to $f(x)$ whenever the sequence $\{x_n\}$ converges to $x$. This no longer characterizes continuity in the more general framework of topological spaces. Instead, we are forced to introduce either *nets* or *filters*. A net is like a sequence, except that instead of being indexed by the natural numbers, the index set can be much larger. Two particularly important techniques for indexing nets include indexing the net by the family of neighborhoods of a point, and indexing the net by the class of all finite subsets of a set.

There are offsetting advantages to working with general topological spaces. For instance, we can define topologies to make our favorite functions continuous. These are called *weak* topologies. The topology of pointwise convergence is actually a weak topology, and weak topologies are fundamental to understanding the equilibria of economies with an infinite dimensional commodity space.

Another important topological notion is compactness. Compact sets can be approximated arbitrarily well by finite subsets. (In Euclidean spaces, the compact sets are the closed and bounded sets.) Two of the most important theorems in this chapter are the Weierstrass Theorem 2.35, which states that continuous functions achieve their maxima on compact sets, and the Tychonoff Product Theorem 2.61, which asserts that the product of compact sets is compact in the product topology (the topology of pointwise convergence). This latter result is the basis of the Alaoglu Theorem 5.105, which describes a general class of compact sets in infinite dimensional spaces.

Liberating the notions of neighborhood and convergence from their metric space setting often leads to deeper insights into the structure of approximation methods. The idea of weak convergence and the keystone Tychonoff Product Theorem are perhaps the most important contributions of general topology to analysis—although at least one of us has heard the complaint that "topology is killing analysis." We collect a few fundamental topological definitions and results here. In the interest of brevity, we have included only material that we use later on, and have neglected other important and potentially useful results. We present no discussion of algebraic or differential topology, and have omitted discussion of quotient topologies, projective and inductive limits, metrizability theorems, extension theorems, and a variety of other topics. For more detailed treatments of general topology, there are a number of excellent standard references, including Dugundji [106], Kelley [198], Kuratowski [218], Munkres [256], and Willard [342]. Willard's historical notes are especially thorough.