Chapter 3

Metrizable spaces

In Chapter 2 we introduced topological spaces to handle problems of convergence that metric spaces could not. Nevertheless, every sane person would rather work with a metric space if they could. The reason is that the metric, a real-valued function, allows us to analyze these spaces using what we know about the real numbers. That is why they are so important in real analysis. We present here some of the more arcane results of the theory of metric spaces. Most of this material can be found in some form in K. Kuratowski's [218] tome. Many of these results are the work of Polish mathematicians in the 1920s and 1930s. For this reason, a complete separable metric space is called a *Polish space*.

Here is a guide to the major points of interest in the territory covered in this chapter. The distinguishing features of the theory of metric spaces, which are absent from the general theory of topology, are the notions of *uniform continuity* and *completeness*. These are not topological notions, in that there may be two *equivalent* metrics inducing the same topology, but they may have different uniformly continuous functions, and one may be complete while the other isn't. Nevertheless, if a topological space is *completely metrizable*, there are some topological consequences. One of these is the Baire Category Theorem 3.47, which asserts that in a completely metrizable space, the countable intersection of open dense sets is dense. Complete metric spaces are also the home of the Contraction Mapping Theorem 3.48, which is one of the fundamental theorems in the theory of dynamic programming (see the book by N. Stokey, R. E. Lucas, and E. C. Prescott [322].)

Lemma 3.23 embeds an arbitrary metric space in the Banach space of its bounded continuous real-valued functions. This result is useful in characterizing complete metric spaces. By the way, all the Euclidean spaces are complete.

In a metric space, it is easy to show that second countability and separability are equivalent (Lemma 3.4). The Urysohn Metrization Theorem 3.40 asserts that every second countable regular Hausdorff is metrizable, and that this property is equivalent to being embedded in the *Hilbert cube*. This leads to a number of properties of separable metrizable spaces. Another useful property is that in metric spaces, a set is compact if and only if it is sequentially compact (Theorem 3.28).

We also introduce the compact metric space called the *Cantor set*. It can be viewed as a subset of the unit interval, but every compact metric space is the image

of the Cantor set under a continuous function. In the same vein, we study the *Baire space* of sequences of natural numbers. It is a Polish space, and every Polish space is a continuous image of it. It is also the basis for the study of *analytic sets*, which we describe in Section 12.5.

We also discuss topologies for spaces of subsets of a metric space. The most straightforward way to topologize the collection of nonempty closed subsets of a metric space is through the Hausdorff metric. Unfortunately, this technique is not topological. That is, the topology on the space of closed subsets may be different for different compatible metrics on the underlying space (Example 3.86). However, restricted to the compact subsets, the topology is independent of the compatible metric (Theorem 3.91). Since every locally compact separable metrizable space has a metrizable compactification (Corollary 3.45), for this class of spaces there is a nice topological characterization of the *topology of closed convergence* on the space of closed subsets (Corollary 3.95). Once we have a general method for topologizing subsets, our horizons are greatly expanded. For example, since binary relations are just subsets of Cartesian products, they can be topologized in a useful way; see A. Mas-Colell [240]. As another example, F. H. Page [268] uses a space of sets in order to prove the existence of an optimal incentive contract.

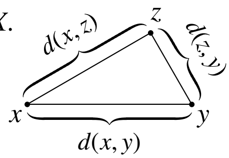
Finally, we conclude with a discussion of the space $C(X, Y)$ of continuous functions from a compact space into a metrizable space under the topology of uniform convergence. It turns out that this topology depends only on the topology of Y and not on any particular metric (Lemma 3.98). The space $C(X, Y)$ is complete if Y is complete, and separable if Y is separable; see Lemmas 3.97 and 3.99.

3.1 Metric spaces

Recall the following definition from Chapter 2.

3.1 Definition *A **metric** (or **distance**) on a set X is a function $d\colon X \times X \to \mathbb{R}$ satisfying the following four properties:*

1. *Positivity: $d(x, y) \geqslant 0$ and $d(x, x) = 0$ for all $x, y \in X$.*



2. *Discrimination: $d(x, y) = 0$ implies $x = y$.*
3. *Symmetry: $d(x, y) = d(y, x)$ for all $x, y \in X$.*
4. *The Triangle Inequality: $d(x, y) \leqslant d(x, z) + d(z, y)$ for all $x, y, z \in X$.*

*A **semimetric** on X is a function $d\colon X \times X \to \mathbb{R}$ satisfying (1), (3), and (4). Obviously, every metric is a semimetric. If d is a metric on a set X, then the pair (X, d) is called a **metric space**, and similarly if d is a semimetric, then (X, d) is a **semimetric space**.*