Chapter 4

Measurability

A major motivation for studying measurable structures is that they are at the foundations of probability and statistics. Suppose we wish to assign probabilities to various *events*. Given events A and B it is natural to consider the events "A and B," "A or B," and the event "not A." If we model events as sets of *states of the world*, then the family of events should be closed under intersections, unions, and complements. It should also include the set of all states of the world. Such a family of sets is called an *algebra* of sets. If we also wish to discuss the "law of averages," which has to do with the average behavior over an infinite sequence of trials, then it is useful to add closure under countable intersections to our list of desiderata. An algebra that is closed under countable intersections is a *σ-algebra*. A set equipped with a σ-algebra of subsets is a *measurable space* and elements of this σ-algebra are called *measurable sets*. In Chapter 10, we discuss the measurability of sets with respect to a *measure*. In that chapter, we show that a measure μ induces a σ-algebra of μ-measurable sets. The reason we do not start with a measure here is that in statistical decision theory events have their own interpretation independent of any measure, and since probability is a purely subjective notion, there is no "correct" measure that deserves special stature in defining measurability.

The first part of this chapter deals with the properties of algebras, σ-algebras, and the related classes of *semirings*, *monotone classes*, and *Dynkin systems*. This means that the ratio of definitions to results is uncomfortably high in this chapter, but these concepts are necessary. The major result in this area is Dynkin's Lemma 4.11. Semirings are important because the class of *measurable rectangles* in a product of measurable spaces is a semiring (Lemma 4.42). The σ-algebra generated by the collection of measurable rectangles is called the *product σ-algebra*.

When the underlying space has a topological structure, we may wish all the open and closed sets to be measurable. The smallest σ-algebra of sets that contains all open sets is called the *Borel σ-algebra* of the topological space. Corollaries 4.15, 4.16, and 4.17 give other characterizations of the Borel algebra. Unless otherwise specified, we view every topological space as measurable space where the σ-algebra of measurable sets is the Borel σ-algebra. The product σ-algebra of two Borel σ-algebras is the Borel σ-algebra of the product topology provided both spaces are second countable (Theorem 4.44).

A function between measurable spaces is a *measurable function* if for every measurable set in its range, the inverse image is a measurable set in the domain. (In probability theory, real-valued measurable functions are known as *random variables*.) Section 4.5 deals with properties of measurable functions: A measurable function from a measurable space into a second countable Hausdorff space (with its Borel σ-algebra) has a graph that is measurable in the product σ-algebra (Theorem 4.45). When the range space is the set of real numbers (with the Borel σ-algebra), the class of measurable functions is a vector lattice of functions closed under pointwise limits of sequences (Theorem 4.27). (It is not generally closed under pointwise limits of nets.) If the range space is metrizable, then the class of measurable functions is closed under pointwise limits (Lemma 4.29). Also, when the range is separable and metrizable, a function is measurable if and only if it is the pointwise limit of a sequence of *simple* measurable functions. This result cannot be generalized too far. Example 4.31 presents a pointwise convergent sequence of Borel measurable functions from a compact metric space (the unit interval) into a compact (nonmetrizable) Hausdorff space whose limit is not Borel measurable. For separable metrizable spaces, the class of bounded Borel measurable real functions is obtained by taking monotone limits of bounded continuous real functions (Theorem 4.33).

A *Carathéodory function* is a function from the product of a measurable space S and a topological space X into a topological space Y that is measurable in one variable and continuous in the other. If the topological spaces are metrizable, then under certain conditions a Carathéodory function is *jointly measurable*, that is, measurable with respect to the product σ-algebra on $S \times X$ (Theorem 4.51). Under stronger conditions (Theorem 4.55) Carathéodory functions characterize the measurable functions from S to $C(X, Y)$ (continuous functions from X to Y).

For Polish spaces, there are some remarkable results concerning Borel sets that are related to the Baire space $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$. Given a Polish space and a Borel subset, there is a stronger Polish topology (generating the same Borel σ-algebra) for which the given Borel set is actually closed (Lemma 4.56). Similarly given a Borel measurable function from a Polish space into a second countable space there is a stronger Polish topology (generating the same Borel σ-algebra) for which the given function is actually continuous. This means that for many proofs we may assume that a Borel set is actually closed or that a Borel measurable function is actually continuous. We use this technique to show every Borel subset of a Polish space is the *one-to-one* continuous image of a closed subset of $\mathcal{N}$ (Theorem 4.60).

It is easy to see that every function f into a measurable space defines a smallest σ-algebra $\sigma(f)$ on its domain for which it is measurable. Theorem 4.41 asserts that a real-valued function is $\sigma(f)$-measurable if and only if it can be written as a function of f. It is also easy to see that every continuous function between topological spaces is Borel measurable (Corollary 4.26). But what is the smallest σ-algebra for which every continuous function is measurable? In general, this σ-algebra is smaller than the Borel σ-algebra, and is called the *Baire σ-algebra*.