Chapter 5

Topological vector spaces

One way to think of functional analysis is as the branch of mathematics that studies the extent to which the properties possessed by finite dimensional spaces generalize to infinite dimensional spaces. In the finite dimensional case there is only one natural linear topology. In that topology every linear functional is continuous, convex functions are continuous (at least on the interior of their domains), the convex hull of a compact set is compact, and nonempty disjoint closed convex sets can always be separated by hyperplanes. On an infinite dimensional vector space, there is generally more than one interesting topology, and the topological dual, the set of continuous linear functionals, depends on the topology. In infinite dimensional spaces convex functions are not always continuous, the convex hull of a compact set need not be compact, and nonempty disjoint closed convex sets cannot generally be separated by a hyperplane. However, with the right topology and perhaps some additional assumptions, each of these results has an appropriate infinite dimensional version.

Continuous linear functionals are important in economics because they can often be interpreted as prices. Separating hyperplane theorems are existence theorems asserting the existence of a continuous linear functional separating disjoint convex sets. These theorems are the basic tools for proving the existence of efficiency prices, state-contingent prices, and Lagrange multipliers in Kuhn–Tucker type theorems. They are also the cornerstone of the theory of linear inequalities, which has applications in the areas of mechanism design and decision theory. Since there is more than one topology of interest on an infinite dimensional space, the choice of topology is a key modeling decision that can have economic as well as technical consequences.

The proper context for separating hyperplane theorems is that of linear topologies, especially locally convex topologies. The classic works of N. Dunford and J. T. Schwartz [110, Chapter V], and J. L. Kelley and I. Namioka, *et al.* [199], as well as the more modern treatments by R. B. Holmes [166], H. Jarchow [181], J. Horváth [168], A. P. Robertson and W. J. Robertson [287], H. H. Schaefer [293], A. E. Taylor and D. C. Lay [330], and A. Wilansky [341] are good references on the general theory of linear topologies. R. R. Phelps [278] gives an excellent treatment of convex functions on infinite dimensional spaces. For applications to prob-

lems of optimization, we recommend J.-P. Aubin and I. Ekeland [23], I. Ekeland and R. Temam [115], I. Ekeland and T. Turnbull [116], and R. R. Phelps [278].

Here is the road map for this chapter. We start by defining a *topological vector space* (tvs) as a vector space with a topology that makes the vector operations continuous. Such a topology is *translation invariant* and can therefore be characterized by the neighborhood base at zero. While the topology may not be metrizable, there is a base of neighborhoods that behaves in some ways like the family of balls of positive radius (Theorem 5.6). In particular, if V is a neighborhood of zero, it includes another neighborhood W such that $W + W \subset V$. So if we think of V as an ε-ball, then W is like the $\varepsilon/2$-ball.

There is a topological characterization of finite dimensional topological vector spaces. (Finite dimensionality is an algebraic, not topological property.) A Hausdorff tvs is finite dimensional if and only if it is locally compact (Theorem 5.26). There is a unique Hausdorff linear topology on any finite dimensional space, namely the Euclidean topology (Theorem 5.21). Any finite dimensional subspace of a Hausdorff tvs is closed (Corollary 5.22) and *complemented* (Theorem 5.89) in locally convex spaces.

There is also a simple characterization of metrizable topological vector spaces. A Hausdorff tvs is metrizable if and only if there is a countable neighborhood base at zero (Theorem 5.10).

Without additional structure, these spaces can be quite dull. In fact, it is possible to have an infinite dimensional metrizable tvs where zero is the only continuous linear functional (Theorem 13.31). The additional structure comes from convexity. A set is *convex* if it includes the line segments joining any two of its points. A real function f is convex if its epigraph, $\{(x, \alpha) : \alpha \geqslant f(x)\}$, is convex. All linear functionals are convex. A convex function on an open convex set is continuous if it is bounded above on a neighborhood of a point (Theorem 5.43). Thus linear functions are continuous if and only if they are bounded on a neighborhood of zero. When zero has a base of convex neighborhoods, the space is *locally convex*. These are the spaces we really want. A convex neighborhood of zero gives rise to a convex homogeneous function known as its *gauge*. The gauge function of a set tells for each point how much the set must be enlarged to include it. In a normed space, the *norm* is the gauge of the unit ball. Not all locally convex spaces are normable, but the family of gauges of symmetric convex neighborhoods of zero, called *seminorms*, are a good substitute. The best thing about locally convex spaces is that they have lots of continuous linear functionals. This is a consequence of the seemingly innocuous Hahn–Banach Extension Theorem 5.53. The most important consequence of the Hahn–Banach Theorem is that in a locally convex space, there are *hyperplanes* that strictly separate points from closed convex sets that don't contain them (Corollary 5.80). As a result, every closed convex set is the intersection of all closed *half spaces* including it.

Another of the consequences of the Hahn–Banach Theorem is that the set of continuous linear functionals on a locally convex space separates points. The