Chapter 6

Normed spaces

This chapter studies some of the special properties of normed spaces. All finite dimensional spaces have a natural norm, the Euclidean norm. On a finite dimensional vector space, the Hausdorff linear topology the norm generates is unique (Theorem 5.21). The Euclidean norm makes $\mathbb{R}^n$ into a complete metric space. A normed space that is complete in the metric induced by its norm is called a *Banach space*. Here is an overview of some of the more salient results in this chapter.

The norm topology on a vector space X defines a topological dual X', giving rise to a natural dual pair $\langle X, X' \rangle$. Thus we may refer to the weak topology on a normed space without specifying a dual pair. In such cases, it is understood that X is paired with its norm dual. Since a finite dimensional space has only one Hausdorff linear topology, the norm topology and the weak topology must be the same. This is not true in infinite dimensional normed spaces. On an infinite dimensional normed space, the weak topology is strictly weaker than the norm topology (Theorem 6.26). The reason for this is that every basic weak neighborhood includes a nontrivial linear subspace—the intersection of the kernels of a finite collection of continuous linear functionals. This linear subspace is of course unbounded in norm, so no norm bounded set can be weakly open (Corollary 6.27). This fact leads to some surprising conclusions. For instance, in an infinite dimensional normed space, zero is always in the weak closure of the unit sphere $\{x : \|x\| = 1\}$ (Corollary 6.29). In fact, in infinite dimensional normed spaces, there always exist nets converging weakly to zero, but wandering off to infinity in norm (Lemma 6.28). Also, the weak topology on an infinite dimensional normed space is never metrizable (Theorem 6.26). Despite this, it is possible for the weak topology to be metrizable when restricted to bounded subsets, such as the unit ball (Theorems 6.30 and 6.31). It also turns out that on a normed space, there is no stronger topology with the same dual. That is, the norm topology is the Mackey topology for the natural dual pair (Theorem 6.23).[1]

[1] The natural duality of a normed space with its norm dual is not always the most useful pairing. Two important examples are the normed spaces $B_b(X)$ of bounded Borel measurable functions on a metrizable space, and the space $L_\infty(\mu)$ of μ-essentially bounded functions. (Both include ℓ_∞ as a special case.) The dual of B_b is the space of bounded charges, but the pairing $\langle B_b, ca \rangle$ of B_b with countably additive measures is more common. See Section 14.1 for a discussion of this pair. Similarly, the dual of L_∞ is larger than L_1, but the pairing $\langle L_\infty, L_1 \rangle$ is more useful. This can be confusing at times.

Linear operators are linear functions from one vector space into another. An important special case is when the range is the real line, which is a Banach space under the absolute value norm. Norms on the domain and the range allow us to define the boundedness of an operator. An operator is *bounded* if it maps norm bounded sets into norm bounded sets. Boundedness is equivalent to norm continuity of an operator, which is equivalent to uniform continuity (Lemmas 5.17 and 6.4). The Open Mapping Theorem 5.18 shows that if a bounded operator between Banach spaces is surjective, then it carries open sets to open sets. The *operator norm* of a bounded operator $T\colon X \to Y$ is defined by $\|T\| = \sup\{\|T(x)\| : \|x\| \leqslant 1\}$. This makes the vector space $L(X,Y)$ of all continuous linear operators from X into Y a normed space. It is a Banach space if Y is (Theorem 6.6). In particular, the topological dual of a normed space is also a Banach space. The Uniform Boundedness Principle 6.14 says that a family of bounded linear operators from a Banach space to a normed space is bounded in the norm on $L(X,Y)$ if and only if it is a pointwise bounded family. This is used to prove that for general dual pairs, all consistent topologies have the same bounded sets (Theorem 6.20).

There are many ways to recognize the continuity of a linear operator between normed spaces. One of these is via the Closed Graph Theorem 5.20, which states that a linear operator between Banach spaces is continuous if and only if its graph is closed. Another useful fact is that a linear operator is continuous in the norm topology if and only it is continuous in the weak topology (Theorem 6.17). Any pointwise limit of a sequence of continuous linear operators on a Banach space is a continuous operator (Corollary 6.19). Every operator T from X to Y, defines an *(algebraic) adjoint operator* T^* from Y^* to X^* by means of the formula $T^*y^* = y^* \circ T$, where X^* and Y^* are the algebraic duals of X and Y respectively. A useful result is that an operator T is continuous if and only if its adjoint carries Y' into X' (Theorem 6.43). Finally, we point out that the evaluation duality $\langle x, x'\rangle$, while jointly norm continuous, is not jointly weak-weak$*$ continuous for infinite dimensional spaces (Theorems 6.37 and 6.38).

The topological dual of a normed space is a Banach space under the operator norm. Alaoglu's Compactness Theorem 6.21 asserts that the unit ball in the dual of a normed space is weak$*$ compact. Since the dual X' of a normed space X is a Banach space, its dual X'' is a Banach space too, called the *second dual* of X. In general, there is a natural isometric embedding of X as a $\sigma(X'', X')$-dense subspace of X'' (Theorem 6.24), and in some cases the two coincide. In this case we say that X is *reflexive*. A Banach space is reflexive if and only if its closed unit ball is weakly compact (Theorem 6.25).

There are some useful results about weak compactness in normed spaces. Recall that for any metric space, a set is compact if and only if it is sequentially com-

For instance, the Mackey topology $\tau(\ell_\infty, \ell_1)$ for the dual pair $\langle \ell_\infty, \ell_1\rangle$ is *not* the norm topology on ℓ_∞: it is weaker. In this chapter at least, we do not deal with other pairings. But when it comes to applying these theorems, make sure you know your dual.