Chapter 7

Convexity

This chapter provides an introduction to *convex analysis*, the properties of convex sets and functions. We start by taking the convexity of the epigraph to be the definition of a convex function, and allow convex functions to be extended-real valued. Any real-valued convex function on a convex set can be extended to the entire vector space by setting it to ∞ where it was previously undefined. The set of points where a convex function does not assume the value ∞ is its *effective domain*. If the effective domain is not empty and the convex function does not assume the value $-\infty$, then it is a *proper* convex function.

By Theorem 5.98 the collection of closed convex sets is the same for all topologies consistent with a given dual pair. Consequently, if a convex function is lower semicontinuous in one consistent topology, then it is lower semicontinuous in every consistent topology. If a convex function is continuous on its effective domain, and the domain is closed, then its extension is lower semicontinuous everywhere. Thus lower semicontinuous proper convex functions are especially interesting. A lower semicontinuous proper convex function is the pointwise supremum of the continuous *affine functions* it dominates (Theorem 7.6).

One of the main themes of this chapter is the maximization of linear functions over subsets of a locally convex space. This is also a recurring theme in economics, where linear functionals are interpreted as prices, and profit maximization and cost minimization are key concepts. The *support functional* of a set assigns to each continuous linear functional its supremum over the set. This supremum may be ∞, which is a prime motive for allowing convex functions to be extended valued. The support functional of any set and its closed convex hull are identical. Since a closed convex subset of a locally convex space is the intersection of the closed half spaces that include it, it is characterized by its support functionals, which encapsulates this information. Thus there is a one-to-one correspondence between closed convex sets and support functionals. Convex sets are partially ordered by inclusion and support functions are ordered pointwise, and the correspondence between them preserves the order structure. See Theorems 7.52 and 7.51 and the following discussion. Even the Hausdorff metric on the space of closed convex subsets of a normed space can be defined in terms of support functionals (Lemma 7.58).

Points at which a nonzero linear functional attains a maximum over a set are *support points* of the set. The associated hyperplane on which the support point lies is called a *supporting hyperplane*. The support point is *proper* if the set does not lie wholly in the supporting hyperplane. Support points must be boundary points, but not every boundary point need be a support point, even for closed convex sets. Indeed, Example 7.9, which is due to V. Klee, provides an example of a nonempty closed convex set in an infinite dimensional Fréchet space that has no support points whatsoever. (In other words, no nonzero continuous linear functional attains a maximum on this set.) However, there are important cases for which support points are plentiful. If a closed convex set has a nonempty interior, then every boundary point is a proper support point (Lemma 7.7). In a finite dimensional space, every point on the *relative boundary* is a proper support point. We also present the Bishop–Phelps Theorem 7.43, which asserts that in a Banach space the set of support points of a closed convex set is a dense subset of the boundary.

We already remarked that a lower semicontinuous convex function is the pointwise supremum of the affine functions it dominates. If it agrees with one of these affine functions at some point, then the graph of the affine function is a supporting hyperplane to the epigraph (Lemma 7.11). The linear functional defining the affine functional is called a *subgradient*. It is easy to see that a convex function attains a minimum at a point only if the zero functional is a subgradient (Lemma 7.10). The collection of subgradients of a convex function at a point in the effective domain is a (possibly empty) weak$*$ compact convex set, called the *subdifferential*. One reason for this terminology is that the *one-sided directional derivative* of a convex function defines a positively homogeneous convex functional, and the set of linear functionals it dominates is the subdifferential (Theorem 7.16). The subgradients of the support functional of a convex set at a particular linear functional in the dual space are the maximizers of the linear functional (Theorem 7.57). The Brøndsted–Rockafellar Theorem 7.50, using an argument similar to the Bishop–Phelps Theorem, shows that in a Banach space, a convex function has a subgradient on a dense subset of its effective domain.

Section 7.5 refines the conditions for the existence of a supporting hyperplane in terms of the existence of cones with particular properties. C. D. Aliprantis, R. Tourky, and N. C. Yannelis [16] provide a survey of their use in economics, where they are called *properness* conditions. A supporting hyperplane is a particular kind of separating hyperplane, so these results also refine our separating hyperplane theorems (Lemma 7.20). In finite dimensional spaces, there are further refinements of the separating hyperplane theorems. In a finite dimensional space, any two nonempty disjoint convex sets can be properly separated (Theorem 7.30). Indeed in a finite dimensional space, two nonempty convex sets can be properly separated if and only if their *relative interiors* are disjoint (Theorem 7.35).

Section 7.6 gives additional properties of proper convex functions on finite dimensional spaces. They are continuous on the relative interiors of their effective