## Chapter 8

# Riesz spaces

A *Riesz space* is a real vector space equipped with a partial order satisfying the following properties. Inequalities are preserved by adding the same vector to each side, or by multiplying both sides by the same positive scalar. Each pair $\{x, y\}$ of vectors has a *supremum* or least upper bound, denoted $x \vee y$. Thus Riesz spaces mimic some of the order properties possessed by the real numbers. However, the real numbers possess other properties not shared by all Riesz spaces, such as order completeness and the Archimedean property. To further complicate matters, the norm of a real number coincides with its absolute value. In more general normed Riesz spaces the norm and absolute value are different.

Riesz spaces capture the natural notion of *positivity* for functions on ordered vector spaces. For the special class of *Banach lattices*, every continuous linear functional is the difference of two positive linear functionals. As a result, many results proven for positive functionals extend to continuous functionals.

The abstraction of the order properties frees them from the details of any particular space and makes it easier to prove general theorems about Riesz spaces in a straightforward fashion. Without this general theory, even special cases are difficult. For example, the well-known Hahn–Jordan and Lebesgue Decomposition Theorems are difficult theorems of measure theory yet are special cases of general results from the theory of Riesz spaces. Conveniently, most spaces used in economic analysis are Riesz spaces, see for instance, [9, 10, 137, 190, 243].

The importance of ordered vector spaces in economic analysis stems from the fact that often there is a natural ordering on commodity vectors for which "more is better." That is, preferences are monotonic in the order on the commodity space. In this case, a reasonable requirement is that equilibrium prices be positive. Furthermore, in Riesz spaces, the order interval defined by the social endowment corresponds roughly to the Edgeworth box. For symmetric Riesz pairs, order intervals are weakly compact, so that the order structure provides a source of compact sets.

This chapter is a brief introduction to the basic theory of Riesz spaces. For a more thorough treatment we recommend C. D. Aliprantis and O. Burkinshaw [12, 15], W. A. J. Luxemburg and A. C. Zaanen [235], P. Meyer-Nieberg [247], H. H. Schaefer [294], and A. C. Zaanen [347].

## 8.1 Orders, lattices, and cones

Recall that a **partially ordered set** $(X, \geqslant)$ is a set $X$ equipped with a partial order $\geqslant$. That is, $\geqslant$ is a transitive, reflexive, antisymmetric relation. The notation $y \leqslant x$ is, of course, equivalent to $x \geqslant y$. Also, $x > y$ means $x \geqslant y$ and $x \neq y$.[1] The expression "$x$ **dominates** $y$" means $x \geqslant y$, and we say "$x$ **strictly dominates** $y$" whenever $x > y$.

Recall that a partially ordered set $(X, \geqslant)$ is a **lattice** if each pair of elements $x, y \in X$ has a supremum (or least upper bound) and an infimum (or greatest lower bound). An element $z$ is the **supremum** of a pair of elements $x, y \in X$ if

i. $z$ is an upper bound of the set $\{x, y\}$, that is, $x \leqslant z$ and $y \leqslant z$; and

ii. $z$ is the least such bound, that is, $x \leqslant u$ and $y \leqslant u$ imply $z \leqslant u$.

The **infimum** of two elements is defined similarly. We denote the supremum and infimum of two elements $x, y \in X$ by $x \vee y$, and $x \wedge y$ respectively. That is,

$$x \vee y = \sup\{x, y\} \quad \text{and} \quad x \wedge y = \inf\{x, y\}.$$

The functions $(x, y) \mapsto x \vee y$ and $(x, y) \mapsto x \wedge y$ are the **lattice operations** on $X$. In a lattice, every finite nonempty set has a supremum and an infimum. If $\{x_1, \dots, x_n\}$ is a finite subset of a lattice, then we write

$$\sup\{x_1, \dots, x_n\} = \bigvee_{i=1}^{n} x_i \quad \text{and} \quad \inf\{x_1, \dots, x_n\} = \bigwedge_{i=1}^{n} x_i.$$

Recall that a subset $C$ of a vector space $E$ is a **pointed convex cone** if:

a. $C$ is a cone: $\alpha C \subset C$ for all $\alpha \geqslant 0$ (equivalently, $\alpha \geqslant 0$ and $x \in C$ imply $\alpha x \in C$);

b. $C$ is convex: which given (a) amounts to $C + C \subset C$ (equivalently, $x, y \in C$ implies $x + y \in C$); and

c. $C$ is pointed: $C \cap (-C) = \{0\}$.

A pointed convex cone $C$ induces a partial order $\geqslant$ on $E$ defined by $x \geqslant y$ whenever $x - y \in C$. The partial order induced by a pointed convex cone $C$ is compatible with the algebraic structure of $E$ in the sense that it satisfies the following two properties:

1. $x \geqslant y$ implies $x + z \geqslant y + z$ for each $z \in E$; and

[1] Note that this notation is at odds with the notation often used by economists for the usual order on $\mathbb{R}^n$, where $x > y$ means $x_i > y_i$ for all $i$, $x \geqq y$ means $x_i \geqslant y_i$ for all $i$, and $x \geqslant y$ means $x \geqq y$ and $x \neq y$.