Chapter 9

Banach lattices

Recall that a lattice norm is norm is a norm that is monotone in the absolute value of a vector (Definition 8.45). Normed Riesz spaces are simply Riesz spaces equipped with lattice norms. By Theorem 8.46, such spaces are locally convex-solid. If the norm is also complete, the space is a *Banach lattice*. Of course, the metric induced by a lattice norm need not be complete, but if it is complete there are surprising consequences. For instance, positive operators between Banach lattices must be continuous. Not every Riesz space can be fitted with a complete lattice norm, but if it can, the norm is unique to positive multiple. A *Fréchet lattice* is a completely metrizable locally solid Riesz space.

In this chapter we start with some examples of Fréchet and Banach lattices and develop some of their basic properties. We continue with a discussion of lattice isometries between Banach lattices and order continuous norms. Of key interest for its wide range of applications is the fact that a Banach lattice and its norm dual form a symmetric Riesz pair if and only if the Banach lattice has order continuous norm. A Banach lattice has *order continuous norm* if every decreasing net that order converges to zero also converges to zero in norm. The other important fact about Fréchet lattices and Banach lattices is that every *positive linear functional* is automatically continuous (Theorem 9.6). Also, for Fréchet lattices the topological and order duals coincide (Theorem 9.11).

We also present, but do not prove, two versions of the Stone–Weierstrass Theorem (Theorems 9.12 and 9.13). These theorems describe dense subspaces of the space of continuous functions a compact space. The lattice version gives conditions for a Riesz subspace to be dense.

There are two important special classes of Banach lattices: the AL-spaces and the AM-spaces. AL-spaces are abstract versions of the $L_1(\mu)$-spaces, while AM-spaces are the abstract versions of the $C(K)$-spaces (K compact Hausdorff). Remarkably, the AL- and AM-spaces are mutually dual. A Banach lattice is an AL-space (resp. an AM-space) if and only if its norm dual is an AM-space (resp. an AL-space). Principal ideals in Banach lattices are the prime examples of AM-spaces. One interesting fact, especially for economists, is that the positive cone of a Banach lattice has nonempty norm interior if and only if it is an AM-space with unit. In AM-spaces, the Stone–Weierstrass Theorem 9.13 provides a plethora of

dense subspaces.

In finite dimensional Euclidean spaces, the positive cone of the space is big in the sense that it has a nonempty interior. In infinite dimensional spaces, the interior of the positive cone of a Banach lattice is often empty. Section 9.6 shows that if the positive cone is nonempty, then the space can be represented as dense subset of a $C(X)$ space.

Next we discuss the properties of positive projections and contractions in Riesz subspaces, and close the chapter with a discussion of the space of functions of bounded variation. This is a space with at least two natural order structures.

9.1 Fréchet and Banach lattices

Recall that a lattice norm $\|\cdot\|$ has the property that $|x| \leqslant |y|$ in E implies $\|x\| \leqslant \|y\|$. A Riesz space equipped with a lattice norm is called a **normed Riesz space**. A complete normed Riesz space is called a **Banach lattice**

9.1 Example (Normed Riesz spaces) Here are some familiar examples of normed Riesz spaces and Banach lattices.

- The Euclidean spaces $\mathbb{R}^n$ with their Euclidean norms are all Banach lattices.

- If K is a compact space, then the Riesz space $C(K)$ of all continuous real functions on K under the sup norm

$$\|f\|_\infty = \sup\{|f(x)| : x \in K\}$$

is a Banach lattice.

- If X is a topological space, then $C_b(X)$, the Riesz space of all bounded real continuous functions on X, under the lattice norm

$$\|f\|_\infty = \sup\{|f(x)| : x \in X\}$$

is a Banach lattice.

- The Riesz space $C[0,1]$ under the L_1 lattice norm

$$\|f\| = \int_0^1 \big|f(x)\big|\,dx$$

is a normed Riesz space, but *not* a Banach lattice.

- If X is an arbitrary nonempty set, then the Riesz space $B(X)$ of all bounded real functions on X under the lattice norm

$$\|f\|_\infty = \sup\{|f(x)| : x \in X\}$$

is a Banach lattice.