2.1 Parent probability distributions, mean value and variance

For a continuous random variable $X$, its probability density function $p(x)$ is defined by

$$
Pr\left[x \leq X \leq x + dx\right] = P(x + dx) - P(x) = \frac{dP(x)}{dx} - \frac{dP(x)}{dx} = p(x)dx
$$

(2.1)

where $P(x)$ is the cumulative probability function, from which it follows that

$$
Pr \left[ X \leq x \right] = P(x) = \int_{-\infty}^{x} p(x)dx
$$

(2.2)

and that $\lim_{x \to \infty} P(x) = 1$. Similarly, for two random variables $X$ and $Y$ the joint probability density function is defined by

$$
p(x,y) = \frac{d^2P(x,y)}{dx\,dy}
$$

(2.3)

where $P(x,y) = Pr\left[X \leq x, Y \leq y\right]$. The mean value and variance of $X$ are given by

$$
\bar{x} = E[X] = \int_{-\infty}^{\infty} x \cdot p(x)dx
$$

$$
Var(X) = \sigma^2_x = E\left[(X - \bar{x})^2\right] = \int_{-\infty}^{\infty} (x - \bar{x})^2 \cdot p(x)dx
$$

(2.4)
Equivalent definitions apply to a discrete random variable $X$. It is in the following assumed that each realisation $X_k$ of $X$ has the same probability of occurrence, and thus, the mean value and variance of $X$ may be estimated from a large data set of $N$ individual realisations:

$$
\bar{x} = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} X_k \\
Var(X) = \sigma_x^2 = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} (X_k - \bar{x})^2
$$

(2.5)

The square root of the variance, $\sigma_x$, is called the standard deviation. Recalling that $E[X] = \bar{x}$, the expression for the variance may be further developed into

$$
\sigma_x^2 = E[(X - \bar{x})^2] = E[X^2 - 2\bar{x}X + \bar{x}^2] = E[X^2] - \bar{x}^2
$$

(2.6)

There are three probability density distributions that are of primary importance in wind engineering. These are the Gaussian (normal), Weibull and Rayleigh distributions, each defined by the following expressions:

$$
p(x) = \frac{1}{\sqrt{2\pi}\sigma_x} \exp\left[-\frac{1}{2}\left(\frac{x - \bar{x}}{\sigma_x}\right)^2\right] \\
p(x) = \beta \frac{x^{\beta-1}}{\gamma^\beta} \exp\left[-\left(\frac{x}{\gamma}\right)^\beta\right] \\
p(x) = \frac{x}{\gamma^2} \exp\left[-\frac{1}{2}\left(\frac{x}{\gamma}\right)^2\right]
$$

(2.7)

They are graphically illustrated in Fig. 2.1. It is seen that a Rayleigh distribution is the Weibull distribution with $\beta=2$. 