Applying Gröbner Bases to Solve IBP Relations

One more approach to solve reduction problems for Feynman integrals is based on the theory of Gröbner bases [56] that have arisen naturally when characterizing the structure of ideals of polynomial rings. The first attempt to apply this theory to Feynman integrals was made in [202,204], where IBP relations were reduced to differential equations. To do this, one assumes that there is a non-zero mass for each line. The typical combination \(a_i 1^+\), where \(1^+\) is a shift operator, is naturally transformed into the operator of differentiation in the corresponding mass. Then one can apply some standard algorithms for constructing corresponding Gröbner bases for differential equations. Another attempt was made in [104] where Janet bases were used.

In this appendix, an approach [180] based on constructing Gröbner bases for polynomials of shift operators is presented. In the next section, Gröbner bases and Buchberger algorithm (as a tool to construct Gröbner bases) in the classical problem of characterizing the structure of ideals of polynomial rings are briefly described. In Sect. F.2, we turn to the approach of [180]. The notion of Gröbner bases is modified, within this approach, in various respects. Examples of applying this approach to solve reduction problems for some families of Feynman integrals are presented in Sect. F.3.

G.1 Gröbner Bases for Ideals of Polynomials

The notion of Gröbner bases was invented by Buchberger [56] when he constructed an algorithm to answer certain questions on the structure of ideals of polynomial rings.

\[1\text{As an application of the method of [202], the solution of the reduction problem for two-loop self-energy diagrams with five general masses was obtained in [204], with an agreement with an earlier solution [201]. Moreover, the solution of the reduction problem for massless two-loop off-shell vertex diagrams (which was first obtained in [40] within Laporta’s algorithm [143,145]) was reproduced in [127].}\]
Let \( A = \mathbb{C}[x_1, \ldots, x_n] \) be the commutative ring\(^2\) of polynomials of \( n \) variables \( x_1, \ldots, x_n \) over \( \mathbb{C} \) and \( I \subset A \) be an ideal\(^3\). A classical problem\(^4\) is to construct an algorithm that shows whether a given element \( g \in A \) is a member of \( I \) or not. A finite set of polynomials in \( I \) is said to be a basis of \( I \) if any element of \( I \) can be represented as a linear combination of its elements, where the coefficients are some elements of \( A \). Let us fix a basis \( \{f_1, f_2, \ldots, f_k\} \) of \( I \). The problem is to find out whether there are polynomials \( r_1, \ldots, r_k \in A \) such that \( g = r_1 f_1 + \ldots + r_k f_k \).

Let \( n = 1 \). In this case any ideal is generated by one element \( f = a_0 + a_1 x + a_2 x^2 + \ldots + a_m x^m \). Now if we want to find out whether an element \( g = b_0 + b_1 x + b_2 x^2 + \ldots + b_l x^l \) can be represented as \( rf \) we first check if \( l \geq m \). If this property holds, we replace \( g \) with \( g - (b_l/a_m)x^{l-m}f \), ‘killing’ the leading term of \( g \). This procedure is nothing but the well-known division of polynomials with a remainder. It is repeated until the degree of a ‘current’ polynomial obtained from \( g \) becomes less than \( m \). It is clear that the resulting polynomial (the remainder) is equal to zero if and only if \( g \) can be represented as \( rf \).

Now let \( n > 1 \). Let us consider an algorithm that will answer this problem for some bases of the ideal. (We will see later that this problem can be solved if we have a so-called Gröbner basis at hand.) To describe it, one needs the notion of an ordering of monomials \( cx_1^{i_1} \ldots x_n^{i_n} \) where \( c \in \mathbb{C} \) and the notion of the leading term (an analogue of the intuitive one in the case \( n = 1 \)). In the simplest variant of the lexicographical ordering, a set \( (i_1, \ldots, i_n) \) is said to be higher than a set \( (j_1, \ldots, j_n) \) if there is \( l \leq n \) such that \( i_1 = j_1, i_2 = j_2, \ldots, i_{l-1} = j_{l-1} \) and \( i_l > j_l \). The ordering is denoted as \( (i_1, \ldots, i_n) \succ (j_1, \ldots, j_n) \). We shall also say that the corresponding monomial \( cx_1^{i_1} \ldots x_n^{i_n} \) is higher than the monomial \( cx_1^{j_1} \ldots x_n^{j_n} \).

One can introduce various orderings, for example, the degree-lexicographical ordering, where \( (i_1, \ldots, i_n) \succ (j_1, \ldots, j_n) \) if \( \sum i_k > \sum j_k \) or \( \sum i_k = \sum j_k \) and \( (i_1, \ldots, i_n) \succ (j_1, \ldots, j_n) \) in the sense of the lexicographical ordering. The only two axioms that the ordering has to satisfy are that 1 is the only minimal element under this ordering and that if \( f_1 \succ f_2 \) then \( gf_1 \succ gf_2 \) for any \( g \).

An ordering can be defined by an ordered set of \( n \) linearly independent combinations

\(^2\)A ring is a set with two operations: multiplication and addition. Associativity and distributivity are usually implied.

\(^3\)A non-empty subset \( I \) of a ring \( R \) is called a left (right) ideal if (i) for any \( a, b \in I \) one has \( a + b \in I \) and (ii) for any \( a \in I, c \in R \) one has \( ca \in I \) (\( ac \in I \) respectively). In the case of commutative rings there is no difference between left and right ideals.

\(^4\)A closely related problem is to find out whether any solution of the equation \( g(x_1, \ldots, x_n) = 0 \) is also a solution of the system of the equations \( f_i(x_1, \ldots, x_n) = 0, i = 1, \ldots, k. \)