Chaos, Periodicity and Complexity on Dynamical Systems

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Summary. In the setting of discrete dynamical systems \((X, f)\) where \(X\) is a compact metric space and \(f\) is a continuous self-mapping of \(X\) into itself, we introduce two ways of appreciating how complicated the dynamics of such systems is. First through several notions of chaos like Li-Yorke and Devaney chaos, sensitive dependence of initial conditions, transitivity, Lyapunov exponents, and the second through different notions of entropy, mainly the Kolmogorov-Sinai and topological entropies. In particular Kolmogorov-Sinai is introduced in a very general way. Also we review some known relations among these notions of chaos and entropies.

Complicated dynamics can be also understood via periodic orbits. To this aim we concentrate in the forcing relations among the periods of the orbits in the simplest cases such that \(I = [0, 1], \mathbb{S}^1\) and in other more complicated spaces. Additionally, we resume some results recently obtained for delay difference equations of the form \(x_{n+k} = f(x_n)\) for \(k \geq 2\).

1.1 Introduction

Roughly speaking we understand by a dynamical system a set of states (called the space of states) evolving with time. More precisely, a dynamical system is a triple \((X, \Phi, G)\) where \(X\) denotes the state space usually given by a topological space, \(\Phi\) is the flow of the system, that is, the rule of evolution, given by a continuous map from \(G \times \mathbb{R}\) into \(X\) and \(G \subseteq \mathbb{R}\) a semigroup of times. When \(G = \mathbb{Z}\) or \(G = \mathbb{Z}^+ \cup 0\) the dynamical system is called discrete and it is denoted by the pair \((X, f)\) where \(X\) is a nonempty metric space and the flow is \(\Phi(n, x) = f^n(x)\) where \(f\) is a continuous map form \(X\) into itself. Given \(X\), we will denote by \(C(X)\) the set of continuous maps from \(X\) into itself. For \(f \in C(X)\) we define its \(n^{th}\)-iterate by \(f^n = f \circ f^{n-1}\), \(n \geq 1\), \(f^0 = \text{identity}\), with \(f \circ g\) denoting the composition of \(f\) and \(g\). When \(G = \mathbb{R}\) the system is called continuous.

The main goal when considering dynamical systems is to understand the long term behavior of states in evolving according with the flow. The systems often involve several variables and are usually nonlinear. In a variety
of settings, very complicated behavior is observed even though the equations themselves describing the system are not very complicated. Thus simple algebraic forms of the equations do not mean that the dynamical behavior is simple; in fact, it can be very complicated or even chaotic. One aspect of the chaotic nature of systems is described by the sensitive dependence on initial conditions which means that initial close states of the system evolve separately.

**Definition 1.1.** The dynamical system \((X, f)\) has sensitive dependence on initial conditions (s.d.i.c.) on \(Y \subseteq X\) if there exists an \(r > 0\) (independent of the points of \(Y\)) such that for each point \(x \in Y\) and for each \(\epsilon > 0\), there exist \(y \in Y\) with \(\rho(x, y) < \epsilon\) and \(n \geq 0\) such that \(\rho(f^n(x), f^n(y)) \geq r\).

One of the first situations where s.d.i.c. appeared was observed by E.Lorenz in his simplified well known system of three differential equation stated as a model for the prediction of the weather. For such systems, if the initial conditions are only approximately specified, then the evolution of the state may be very different. This fact leads to important difficulties when using approximate, or even real, solutions to predict future states based on present knowledge. To develop an understanding of these aspects of chaotic dynamics, we want to find situations which exhibit this behavior and yet for which we can still understand the important features of how solutions evolves with time.

Sometimes we cannot follow a particular solution with complete certainty because there is round off error in the calculations or we are using some numerical scheme to find it. We are interested to know whether the approximate solution we calculate is related to a true solution of the exact equations. In some of the chaotic systems, we can understand how an ensemble of different initial conditions evolves, and prove that the approximate solution traced by a numerical scheme is shadowed by a true solution with some nearby initial conditions. One typical example of such behavior is given by the weather, see [61].

If the system models the weather, people may not be content with the range of possible outcomes of the weather that could develop from the known precision of the previous conditions, or to know that a small change of the previous conditions would have produced the weather which had been predicted. However, even for a subject like weather, for which quantitative as well as qualitative predictions are important, it is still useful to understand what factors can lead to instabilities in the evolution of the state of the system. It is now realized that no new better simulation of weather on more accurate computers of the future will be able to predict the weather more than about fourteen days ahead, because of the very nonlinear nature of the evolution of the state of weather. This type of knowledge can by itself be useful.

In recent years, dynamical systems has had many applications to science and engineering, some of which have gone under the related headings of chaos